Notes on sets and logic

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1 Introduction

These notes on the theory of sets are mainly meant as a skeleton that I can pick apart and use when writing other documents. I do lots of applied math, and lots of applied math teaching, and for various reasons I find myself talking and writing about sets and logic a lot. At a certain point it became clear that I would be better off if I just wrote down some basic stuff about these topics in one place. A secondary goal is to state ideas and rules that I want to be able to quickly look up and remember, and to present examples so that I quickly remember how to apply them.

These notes are based largely on works from the library of Elaine H. Koppelman-Eugster. For this and much else, thank you Elaine. The parts of these notes from before the 2020 quarantine were written by hand at the bar of Ann Arbor's great Fraser's Pub, and only later typed up. Though they neglect to mention this on their website, Fraser's is an excellent place to study set theory.

2 Sets and relations

2.1 Initial definitions

Definition 1. [1: p. 3] A **set** S is any collection of definite, distinguishable objects, to be considered as a whole.

Definition 2. [1: p. 3] The objects in a set are called its **elements** or **members**. If s is a member of S, we denote its membership as $s \in S$.

Definition 3. [1: p. 5] **Equality of sets** (or **The intuitive principle of extension**): Two sets are equal iff they have the same members.

Definition 4. [1: p. 7] **The intuitive principle of abstraction**: A formula $P(x)$ defines a set A by the convention that the members of A are exactly those objects a such that $P(a)$ is a true statement. We may write this is as $\{x|P(x)\}$, read as "the set of all x such that $P(x)$ is true".

Definition 5. [1: p. 10] For two sets A and B, we say that A is **included in** B, denoted $A \subseteq B$, iff each member of A is also a member of B. In this case we also say that A is a **subset** of B.

Definition 6. [1: p. 11] If set A is a subset of set B then we say that B **includes** A, denoted $B \supseteq A$.

Definition 7. [1: p. 11] Set A is **properly included in** set B, denoted $A \subset B$, iff $A \subseteq B$ and $A \neq B$. Then A is also called a **proper subset** of B, and B **properly includes** A.

Definition 8. [1: p. 11] The set containing no elements is called the **empty set** and is denoted $\emptyset$.

Definition 9. [1: p. 12] The set of all subsets of a set A is called the **power set** of A, denoted $\mathcal{P}(A)$.

2.2 Operations

Definition 10. [1: p. 14] The **union** of sets A and B is denoted $A \cup B$, and is defined as $A \cup B \equiv \{x|x \in A \vee x \in B\}$.

Definition 11. [1: p. 14] The **intersection** of sets A and B is denoted $A \cap B$, and is defined as $A \cup B \equiv \{x | x \in A \wedge x \in B\}$.

Definition 12. [1: p. 14] Sets A and B are **disjoint** iff $A \cap B = \emptyset$. If instead $A \cap B \neq \emptyset$, then A and B are said to **intersect**. A collection of sets which are all pairwise disjoint is called a **disjoint collection**.

Definition 13. [1: p. 14] A **partition** of a set X is a disjoint collection $\mathcal{A}$ of nonempty and distinct subsets of X such that each member of X is a member of some (so, exactly one) member of $\mathcal{A}$.

Definition 14. [1: p. 14] The **absolute complement** of a set A, denoted $\overline{A}$, is $\{x | x \notin A\}$.

Definition 15. [1: p. 14] The **relative complement** of a set A with respect to a set X is $X \cap \overline{A}$, which is usually shortened to $X \setminus A$, meaning $\{x \in X | x \notin A\}$.

Definition 16. [1: p. 14] The **symmetric difference** of sets A and B, denoted $A \triangle B$, is defined by $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Definition 17. [1: p. 15] When all sets under consideration are subsets of a set U, then U is called the **universal set**.

2.3 Algebra of sets

Theorem 1. *[1: p. 20] Dual theorems: beginning with a true theorem, if we switch* $\cap$ *with* $\cup$ *and* U *with* $\emptyset$*, then we obtain a deducible theorem. That theorem is called the* ***dual*** *of the original.*

Theorem 2. *[1: p. 20] The* ***principal of duality****: If* T *is any theorem expressed in terms of* $\cup$, $\cap$, *and* $\overline{}$, *then the dual of* T *is also a theorem.*

The following properties, which are always true, are all dual theorems [1: p. 18, 21]:

1. $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$
2. $A \cup B = B \cup A$ and $A \cap B = B \cap A$
3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. $A \cup \emptyset = A$ and $A \cap U = A$

5. $A \cup \overline{A} = U$ and $A \cap \overline{A} = \emptyset$
6. $A \cup B = A\ \forall A \implies B = \emptyset$ and $A \cap B = A\ \forall A \implies B = U$
7. $A \cup B = U$ and $A \cap B = \emptyset \implies B = \overline{A}$
8. $\overline{\overline{A}} = A$
9. $\overline{\emptyset} = U$ and $\overline{U} = \emptyset$
10. $A \cup A = A$ and $A \cap A = A$
11. $A \cup U = U$ and $A \cap \emptyset = \emptyset$
12. $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$
13. $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$

The proofs of the above properties are a bit tedious, the general strategy being to show that LHS $\subseteq$ RHS $\wedge$ RHS $\subseteq$ LHS. This is done by considering a generic element of the LHS and showing that it has membership in the RHS, and then also doing the reverse. Let's do one example.

Proof. [1: p. 18 – 19] First prove that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, so x is in their intersection too. If $x \in B \cap C$, then $x \in B$ and $x \in C$. Hence $x \in A \cup B$ and $x \in A \cup C$, so x is a member of their intersection in this case too.

Next prove that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. Hence, $x \in A$, or $x \in B$ and $x \in C$. This implies that $x \in A \cup (B \cap C)$.

□

Theorem 3. *[1: p. 21] The following statements are equivalent:*

- ↪ $A \subseteq B$
- ↪ $A \cap B = A$
- ↪ $A \cup B = B$

2.4 Relations

It might seem that set theory has a problem represent the fundamental idea of ordered lists, because sets do not encode order. But in set theory we can actually represent an **ordered pair** using a set containing 2 sets. One set specifies which of the elements comes first, and another set specifies which elements are included in the pair (the two sets are always distinguished by their cardinality, since by necessity there is one subset with one element and another subset with two elements). So to represent the ordered pair (x, y) by means of sets, we could use the set $\{\{x\}, \{x, y\}\}$.

Of course, any ordered n-tuple can be built in this fashion: (x, y, z) can be represented as $((x, y), z)$. Because this is an ordered pair with an embedded ordered pair, we can simply apply our set theoretic definition of an ordered pair twice. In this way a definition for any ordered n-tuple can be built with n iterated applications of the ordered pair idea.

From this we can construct a **binary relation**. Take a relation ρ, and consider 2 objects $x, y,$, then if they are related by that relation we write $x\rho y$ and write $(x, y) \in \rho$. The underlying idea is that the relation generates a set of objects which satisfy that relation.

Of course we can do similarly for relations which take more than 2 objects.

Theorem 4. *[1: p. 26] The ordered pair of x and y is uniquely determined by x and y. Moreover, if $(x, y) = (u, v)$, then $x = u$ and $y = v$.*

Definition 18. [1: p. 27] A **binary relation** ρ is a set of ordered pairs.

Definition 19. [1: p. 27] If ρ is a relation, we write $(x, y) \in \rho$ and $x\rho y$ interchangeably, and we say that x is ρ-**related to** y iff $x\rho y$.

Definition 20. [1: p. 28] If ρ is a relation, then the **domain** of ρ, symbolized by D_ρ, is $\{x|$ for some $y, (x, y) \in \rho\}$.

Definition 21. [1: p. 28] If ρ is a relation, then the **range** of ρ, symbolized by R_ρ, is $\{y|$ for some $x, (x, y) \in \rho\}$.

Definition 22. [1: p. 28] The **cartesian product** of sets X and Y, denoted $X \times Y$, is defined as $X \times Y = \{(x, y)|x \in X \wedge y \in Y\}$.

Definition 23. [1: p. 29] The set of points corresponding to the members of a relation is called the **graph** of the relation.

2.5 Equivalence relations

Definition 24. [1: p. 32] A relation ρ in a set X is an **equivalence relation** if it satisfies all of the following 3 properties:

$\hookrightarrow$ $x\rho x \;\; \forall x \in X$ (reflexivity)

$\hookrightarrow$ $x\rho y \implies y\rho x$ (symmetry)

↪ $x\rho y, y\rho z \implies x\rho z$ (transitivity)

Remark 1. [1: p. 32] An equivalence relation "divides the population into disjoint subsets".

Definition 25. [1: p. 33] For ρ an equivalence relation on set X, a subset A of X is an **equivalence class** (or a ρ-equivalence class) iff there is a member x of A such that A is equal to the set of all y for which $x\rho y$. So, A is an equivalence class iff $\exists x \in X$ such that $A = \rho[\{x\}]$.

When we specify a particular relation ρ, call the set of all ρ-relatives of x in X just $[x]$, called the equivalence class generated by x. Two properties: [1: p. 33]

↪ $x \in [x]$

↪ if $x\rho y$, then $[x] = [y]$

Theorem 5. *[1: p. 33] Let ρ be an equivalence relation on X. Then the collection of distinct ρ-equivalence classes is a partition of X. Conversely, if $\mathcal{P}$ is a partition of X, and a relation ρ is defined by $a\rho b$ iff there exists A in $\mathcal{P}$ such that $a, b \in A$, then ρ is an equivalence relation on X. Also, if an equivalence relation ρ determines the partition of $\mathcal{P}$ of X, then the equivalence relation defined by $\mathcal{P}$ is equal to ρ. Conversely, if a partition $\mathcal{P}$ of X determines the equivalence relation ρ, then the partition of X defined by ρ is equal to $\mathcal{P}$.*

Theorem 6. *[1: p. 35] A relation ρ is an equivalence relation iff there exists a disjoint collection $\mathcal{P}$ such that $\rho = \{(x, y)|$ for some C in $\mathcal{P}, (x, y) \in C \times C\}$.*

2.6 Functions

Definition 26. [1: p. 37] A **function** is a relation f that meets the following requirements:

↪ The members of f are ordered pairs

↪ If (x, y) and (x, z) are members of f, then $y = z$

Remark 2. With this wonderful definition we have now built it up so that functions are relations, defined in terms of sets!

Definition 27. [1: p. 38] A function f is **into** Y iff $R_f \subset Y$.

Definition 28. [1: p. 38] A function f is **onto** Y iff $R_f = Y$.

Definition 29. [1: p. 39] A function f is **one-to-one** iff $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$. Often the contrapositive is a more convenient statement: $f(x_1) = f(x_2) \implies x_1 \neq x_2$.

Definition 30. [1: p. 40] If a function $f : X \to Y$ is one-to-one and onto then it is called a **one-to-one correspondence between X and Y**.

Definition 31. [1: p. 38] A function f is **on** X when the domain of f is X.

Definition 32. [1: p. 39] The symbols $f : X \to Y$ and $X \xrightarrow{f} Y$ denote that f is a function on the set X into the set Y.

Definition 33. [1: p. 42] The **composite** of functions f and g, denoted $g \circ f$, is $\{(x, z) | (\exists y | xfy \wedge ygz)\}$. The operation itself is called **composition**.

Remark 3. [1: p. 42] Here is a special case of Definition 33: If $f : X \to Y$ and $g : Y \to Z$, then $g \circ f : X \to Z$ and $(g \circ f)(x) = g(f(x))$.

Definition 34. [1: p. 45] If f is a one-to-one function, the function resulting from interchanging the coordinates of members of f is called the **inverse function** of f. The inverse function is denoted f^{-1}.

Remark 4. [1: p. 45] Note that functional inversion is defined only for one-to-one functions. It is straightforward to show that if we allowed functions which are not one-to-one to have an inverse, they would contain at least one pair which violates the definition of a function.

Remark 5. [1: p. 45] If f^{-1} is the inverse of f then its domain is the range of f, its range is the domain of f, and $x = f^{-1}(y) \iff y = f(x)$.

Theorem 7. *[1: p. 45] If f and g are one-to-one functions, then $g \circ f$ is one-to-one, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

2.7 Ordering relations

Remark 6. [1: p. 47] It's very natural that we want to be able to represent orders, so that we can talk about relations ρ and sets X such that $x \rho y$ for some distinct members $x, y \in X$, but it is not the case that $y \rho x$. This suggests that it would be useful to be able to plce x and y in the order x, y rather than in the order y, x, so that the truthfulness of $x \rho y$ is suggested by the order of the elements. Examples of these types of orderings are "increasing" and "decreasing" sequences, which encode the truthfulness of $x\rho y$ and the falsehood of $y\rho x$ in the ordering y, x, where ρ is given by the ordering relations that we traditionally represent as $\leq$ and $\geq$ respectively. A similar motivation makes us want to create orders according to the relations $\subseteq$ and $\supseteq$.

Definition 35. [1: p. 47] A relation ρ in X is **antisymmetric** iff $x\rho y \wedge y\rho x \implies x = y \ \ \forall x, y \in X$.

Definition 36. [1: p. 48] A **partial ordering** in a set X is a reflexive, symmetric, and transitive relation in X.

Definition 37. [1: p. 48] A relation ρ **partially orders** a set Y iff $\rho \cap (Y \times Y)$ is a partial ordering in Y.

Definition 38. [1: p. 49] An element y of X is a **cover** of x in X iff $x < y$ and there exists no u in X such that $x < u < y$.

Definition 39. [1: p. 49] A relation ρ is a **simple ordering** iff it is a partial ordering such that $x\rho y$ or $y\rho x$ whenever x and y are distinct members of the domain (which is equal to the range) of ρ.

Definition 40. [1: p. 49] A relation ρ **simply orders** a set Y iff $\rho cap(Y \times Y)$ is a simple ordering in Y.

Definition 41. [1: p. 49] A **partially ordered set** is an ordered pair $(X, \leq)$ such that $\leq$ partially orders X.

Definition 42. [1: p. 50] A **simply ordered set** is an ordered pair $(X, \leq)$ such that $\leq$ simply orders X.

Definition 43. [1: p. 51] A function $f : X \to X'$ is **order preserving (isotone)** relative to an ordering $\leq$ for X and an ordering $\leq'$ for X' iff $x \leq y \implies f(x) \leq' f(x)$.

Definition 44. [1: p. 51] An **isomorphism** between the partially ordered sets $(X, \leq)$ and $(X', \leq')$ is a one-to-one correspondence between X and X' such that both it and its inverse are order-preserving.

Remark 7. [1: p. 51] If an isomorphism exists between partially ordered sets $(X, \leq)$ and $(X', \leq')$, then each partially ordered set is an **isomorphic image** of the other, and the two partially ordered sets are called **isomorphic**.

Theorem 8. *[1: p. 52] A partially ordered set $(X, \leq)$ is isomorphic to a collection of sets: a collection of subsets of X, partially ordered by inclusion.*

Definition 45. [1: p. 52] A **least** member of a set X relative to a partial ordering $\leq$ is a y in X such that $y \leq x \ \ \forall x \in X$. A **minimal** member of a set X relative to $\leq$ is a $y \in X$ such that for no x in X is $x < y$.

Definition 46. [1: p. 52] A **greatest** member of a set X relative to a partial ordering $\leq$ is an y in X such that $x \leq y \ \ \forall x \in X$. A **maximal** member of a set X relative to $\leq$ is a $y \in X$ such that for no x in X is $y < x$.

Definition 47. [1: p. 52] A partially ordered set $(X, \leq)$ is **well-ordered** iff each nonempty subset has a least member.

Definition 48. [1: p. 53] If $(X, \leq)$ is a partially ordered set and $A \subseteq X$, then an element $x \in X$ is an **upper bound** for A if, $\forall a \in A$, $a \leq x$. An element $x \in X$ is a **lower bound** for A if, $\forall a \in A$, $x \leq a$. An $x \in X$ is a **least upper bound** or **supremum** of A iff x is an upper bound for A and $x \leq y$ for any upper bound y of A. An $x \in X$ is a **greatest lower bound** or **infimum** of A iff x is a lower bound for A and $x \geq y$ for any lower bound y of A.

References

[1] Robert R. Stoll. *Sets, Logic, and Axiomatic Theories.* W. H. Freeman and Company, 1961.