The Clock Game: Phrasing a valid equality from an arbitrary sequence of digits

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1 The Clock Game

Many bored students who have watched the minutes tick by on a classroom clock will recognize the following game. Consider any sequence S of digits, together with a set $\mathcal{O}$ of allowable binary operations. Using the allowable operations any number of times, placing only one equals sign between any two digits, and always separating the digits with either an operation or an equals sign, the challenge is to create as many true statements as possible. Under appropriate constraints, S represents the sequence on the face of a digital clock, which every new minute provides a fresh opportunity to kill time.

For example, suppose we have allowed ourselves the set $\mathcal{O} = \{+, -\}$, and somehow obtained $S = 123$. Ideally this sequence should appear in the wild. It might be the page you were on when you put down your book, or a Toronto commuter might read it off the Sherway bus at Kipling Station. Then the possible attempts at equalities are as follows:

Equality	Truth value
1 = 2 + 3	0
1 = 2 - 3	0
1 + 2 = 3	1
1 - 2 = 3	0

To make sure we are clear on the rules of the game, consider an example of what is not allowed. Suppose we find the sequence $S = 1110$, and we offer the attempted solution $1 = 1 = 01$. This idea breaks three important rules: we are only permitted to use one equals sign; we cannot reorder the digits of the sequence; and we cannot combine digits to create a new number. If the set $\mathcal{O}$ of allowed operations can be used to translate some sequence S into a valid equality, then we might call S *phraseable* under $\mathcal{O}$. If $\mathcal{O}$ cannot be used to translate S into a valid equality, then S is not phraseable under $\mathcal{O}$.

Which sequences are phraseable, and what are the characteristics of phraseable sequences? To answer these questions, I show results from brute force solutions of all sequences of length 4 that have integer digits from 0 to 9. We might very often encounter sequences that can appear on 24 hour

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clocks. Let's start by considering the four traditional binary operations of arithmetic: addition (+), subtraction (-), multiplication (*), and division (÷), and I first show results from the simplified game in which these operations can only be applied to the number sequence in order from left-to-right. I then also consider the arithmetical operators with any order of operations, as well as exponentiation (^) in addition to the operations of arithmetic.

2 What sequences are phraseable?

Let's pretend that we found a sequence S on the face of a 24 hour clock. So, consider a sequence S of length 4, where the first pair of digits represents an hour and the second pair of digits represents a minute; S can appear on a 24 hour clockface if and only if the first pair of digits does not exceed 23, and the second pair of digits does not exceed 59. The natural question is: which times can be translated into valid equalities under the default set of arithmetical operators? Placing the equals sign between any two digits, and applying any combination of the four standard arithmetical operations $(+, -, *, \div)$ from left to right, is it possible to translate a given time into a valid equality in at least one way?

Figure 1 shows which times are phraseable on a standard 24 hour clockface. To make the figure, I wrote a Python program that checks every combination of hours and minutes to see whether or not some solution exists, and saves the results in a SQL database. Then, I read them into the programming language R. There I generated a 24×60 matrix that has a 1 in every index that corresponds to a phraseable time, and a 0 otherwise. Finally, I replaced every 1 with the hexadecimal code for a dark colour, and 0 with a light colour. Then I used the rect() command in R to add a rectangle to the plot for every hour and minute, and coloured them according to the corresponding colour value in the matrix.

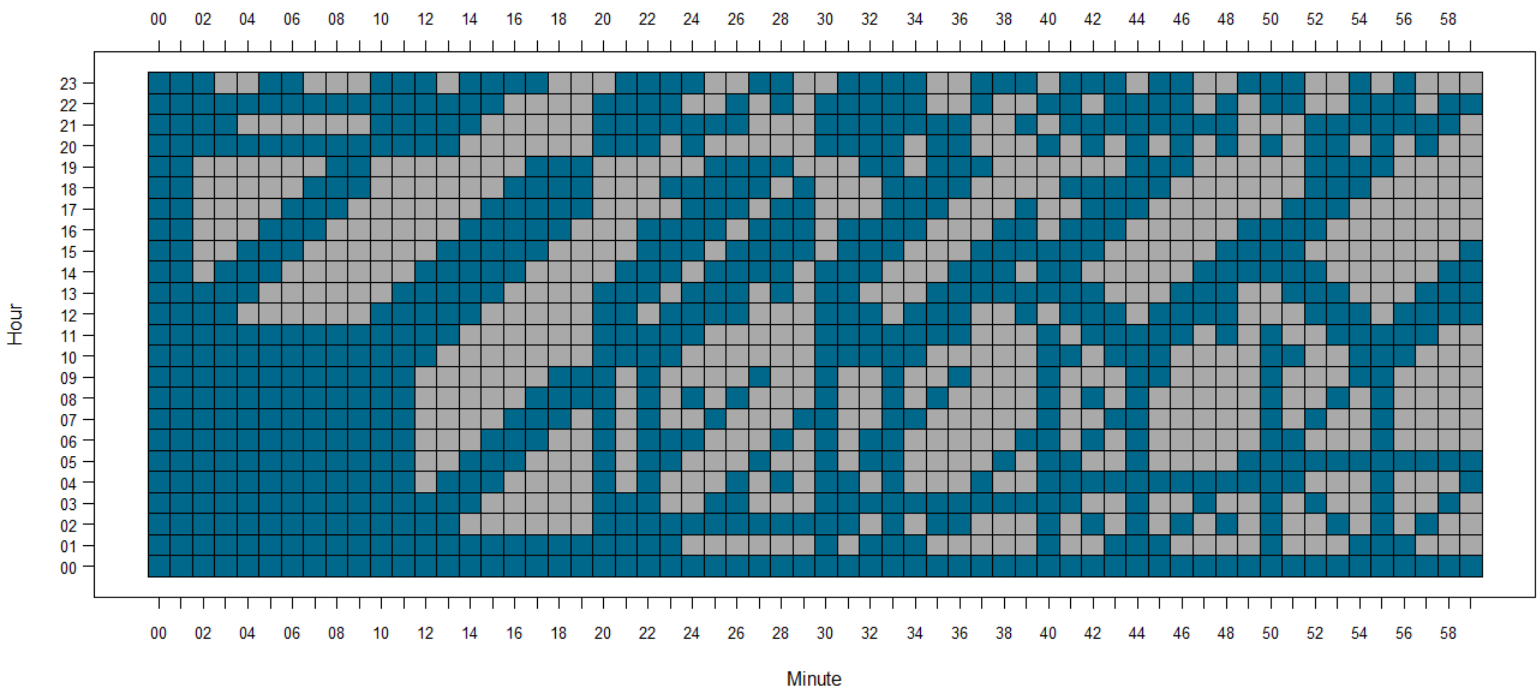


Figure 1: Phraseable times, operations evaluated Left-to-Right, using $\mathcal{O} = \{+, -, *, \div\}$. Each cell is a combination of an hour and a minute on a 24 hour clock. Times which are phraseable (at least one valid equality can be constructed) are blue, and times which are not phraseable are grey. The results are obtained by applying any combination of the operators $\{+, -, *, \div\}$ to a sequence and placing an equals sign anywhere, and evaluating the operations from left to right.

The local regularities in Figure 1 combined with its chaotic overall impression make it reminiscent of certain cellular automata [1], or some almost-regular images of Mandelbrot [2]. However, an enterprising player — say, someone who finishes their dinner and retires to the living room at 19:00 to play our game, only to discover that most of the time they cannot make any valid equalities — might wish to find a way to make Figure 1 less sparse. This inspires two complications: considering the order of operations, and expanding the set of allowed operations.

Permitting operations to be applied in an order other than left to right gives a very natural avenue for increasing the possibility of success in the game. To see why this is appropriate, notice first that the impossibility of some times in this figure appears quite natural, while the impossibility of other times is more surprising. For example, it is likely not surprising that the sequence $S = 1911$ does not give rise to any valid expression using only arithmetical operators. But what about the sequence $S = 1222$? By the strict left-to-right order of operations, there are 48 ways of attempting to form a valid equality from this sequence, and every one of them is false. It would be natural to try to form the statement $1 + 2 \div 2 = 2$, but this is not valid if we are restricted to applying operations from left to right: $1 + 2 = 3$, and $\frac{3}{2} \neq 2$. Instead, we wish to compute $1 + (2 \div 2) = 2$. So, there are good reasons to relax the requirement that operations are applied in order from left to right. To similarly motivate the inclusion of exponentiation, notice two visible patterns in which sequences are not phraseable under the basic left-to-right arithmetical setup. First, numbers with only one zero (mostly those in the bottom-right of Figure 1) are rarely phraseable. Second, numbers which include a mix of small digits and large digits are also rarely phraseable. One operation can conveniently bridge both of these gaps: the binary operation of exponentiation will allow zeroes to be used more often in constructing valid equalities, and can frequently connect small numbers to big numbers in order to make the game more interesting.

Figure 2 shows the phraseable times when any order of operations is permitted.

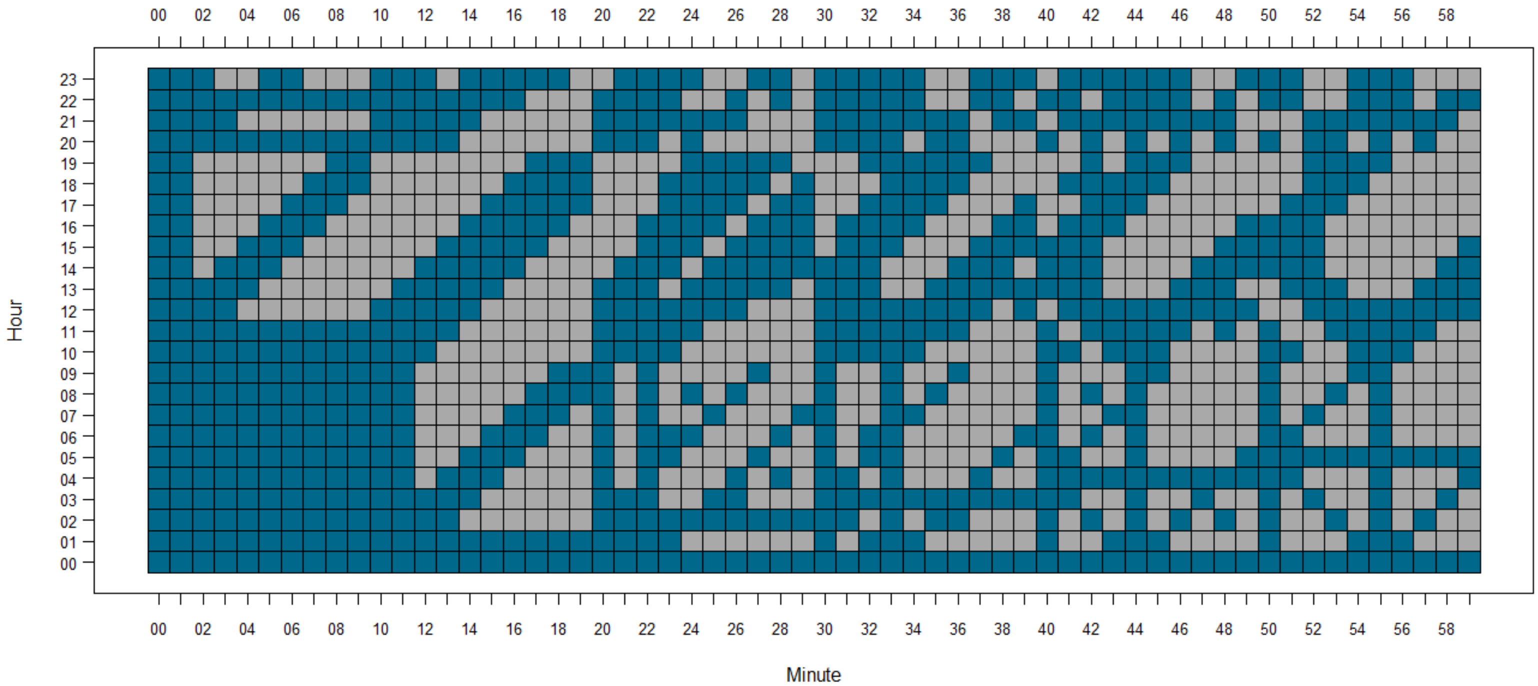


Figure 2: Phraseable times on a 24 hour clockface, operations evaluated in any order, using $\mathcal{O} = \{+, -, *, \div\}$.

Figure 2 only very slightly increases the number of phraseable times (but note that differences do exist; the motivating example, $S = 1222$, is now phraseable). More successful is Figure 3, which shows phraseable times with a left-to-right order of operations and the expanded operation set $\{+, -, *, \div, \hat{}\ \}$. If you're curious about why not use exponentiation with any order of operations, consider the difference between explicitly computing 2^3^5^9 from left to right and computing it with any order of operations.

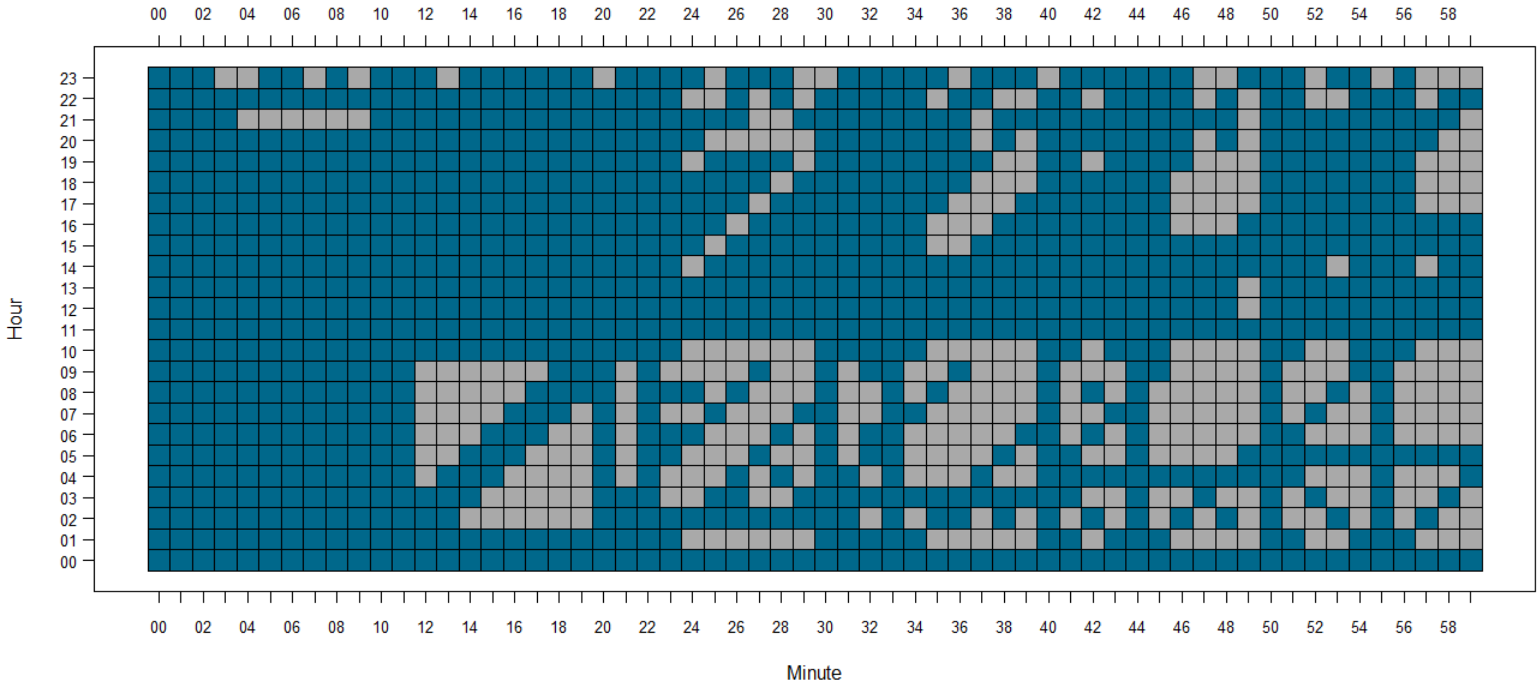
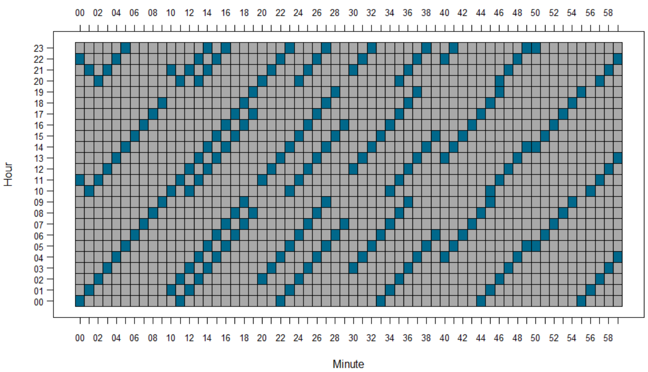
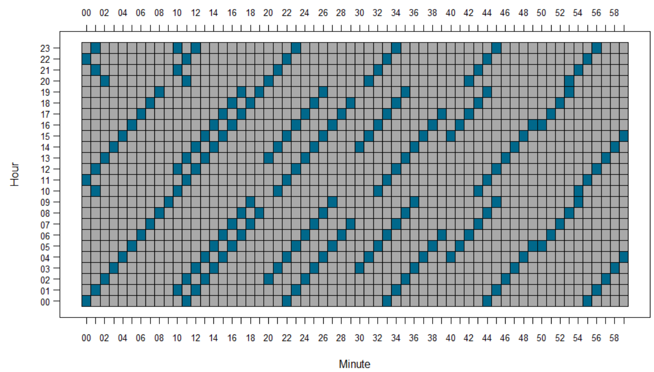


Figure 3: Phraseable times on a 24 hour clockface, operations evaluated Left-to-Right, using $\mathcal{O} = \{+, -, *, \div, ^\wedge\}$. Each cell is a combination of an hour and a minute on a 24 hour clock. Times which are phraseable (at least one valid equality can be constructed) are blue, and times which are not phraseable are grey. The results are obtained by applying any combination of the operators $\{+, -, *, \div, ^\wedge\}$ to a sequence and placing an equals sign anywhere, and evaluating the operations from left to right.

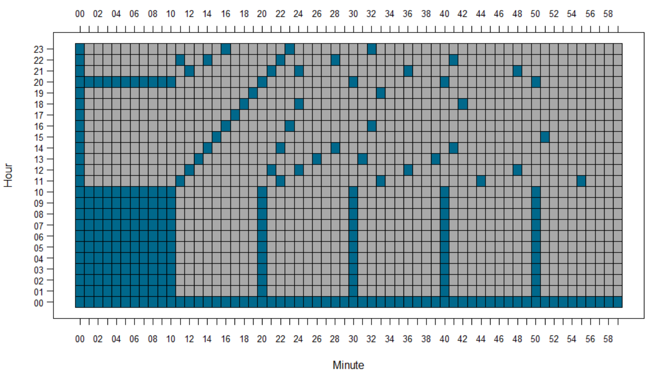
So far, all of these figures have used the full set of permitted operations. But a player might wonder which operations are most likely to yield many phraseable sequences. Figure 4 splits the image into the usefulness of each permitted operation individually, so it considers the cases where the operation set is, respectively, $\mathcal{O} = \{+\}, \mathcal{O} = \{-\}, \mathcal{O} = \{*\}, \mathcal{O} = \{\div\}$, and $\mathcal{O} = \{^\wedge\}$. Note that the preceding images are more than the sum of their parts: times can be phraseable under a large operation set like $\mathcal{O} = \{+, -, *, \div, ^\wedge\}$ which are not phraseable under any one of the sets $\mathcal{O} = \{+\}, \mathcal{O} = \{-\}, \mathcal{O} = \{*\}, \mathcal{O} = \{\div\}$, or $\mathcal{O} = \{^\wedge\}$. One such example is $S = 2258$, which requires a combination of these operations in order to be phraseable: The unique valid equality for $S = 2258$ under $\mathcal{O} = \{+, -, *, \div, ^\wedge\}$ is $2 = 2 * 5 - 8$, and the operations must be evaluated from left to right. However, any time which is phraseable under one operation must be phraseable under any set $\mathcal{O}$ which includes that operation, since we could simply pick that operation repeatedly from $\mathcal{O}$.



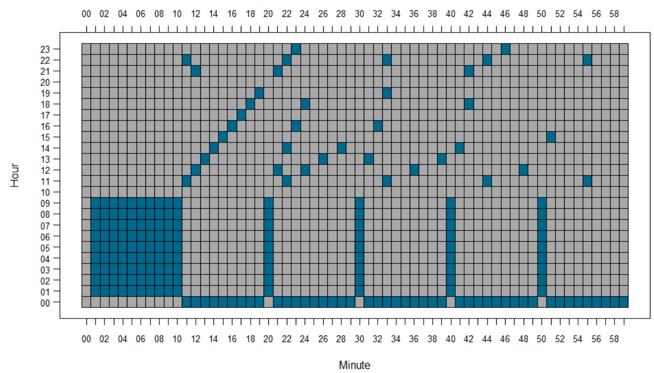
(a) $\mathcal{O} = \{+\}$



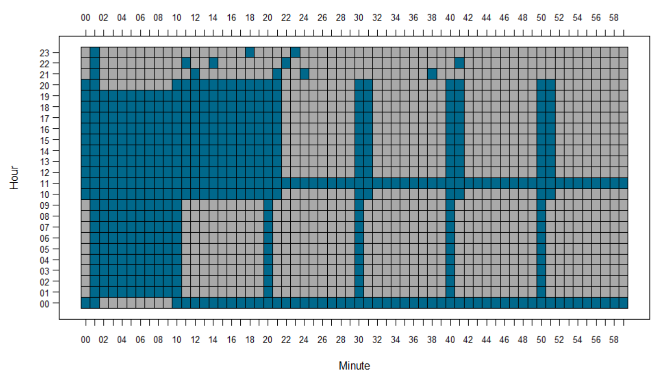
(b) $\mathcal{O} = \{-\}$



(c) $\mathcal{O} = \{*\}$



(d) $\mathcal{O} = \{\div\}$



(e) $\mathcal{O} = \{\wedge\}$

Figure 4: Phraseable times on a 24 hour clockface, operations evaluated Left-to-Right, by each operation. Each cell is a combination of an hour and a minute on a 24 hour clock. Times which are phraseable (at least one valid equality can be constructed) are blue, and times which are not phraseable are grey. The results are obtained by applying only one of the operators $\{+, -, *, \div, \wedge\}$ to a sequence and placing an equals sign anywhere, and evaluating the operations from left to right.

In Figure 4, we begin to see how the chaotic patterns in the previous figures have arisen, although clearly the combination of all operations is far more than the composite of the subfigures in Figure 4. Most notably, many sequences of large numbers (in the top-right of Figure 3) become phraseable only when a combination of operators are permitted. One notable high-level pattern is that, with few exceptions, the operations $+$ and $-$ enable long diagonal stripes of phraseable sequences where the differences or sums of numbers can be made equal, whereas the operations $*$, $\div$, and ^ provide either vertical or horizontal stripes of solutions, or large contiguous blocks of them.

3 Further Ideas

- What are the best binary operations to add? Is there a common binary operation not included here that makes all sequences containing a particular combination of numbers phraseable?
- Can you modify the game to include other types of operations? For example, what happens if I am allowed to take the limit of a sequence in which I apply a unary operation to one digit n times, as $n \to \infty$?
- What if instead of insisting on exact equalities, we take a sequence, place an inequality between any two digits, and attempt to obtain the smallest difference possible? Or, try to obtain a disastrously immense difference?
- This problem is similar to Crazy Sequential Representations [3, 4]. Take a look at those papers. How are Crazy Sequential Representations different from the clock game?
- If you liked this game, you'll like the calculations that Rachel Riley does on the television program *Countdown*.

References

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[2] Mandelbrot, B. B. (1983). *The Fractal Geometry of Nature*. W. H. Freeman and Company.

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