Using Agent-Based Models to Simulate Strategic Behavior in Elections*

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July 10, 2019

*Prepared for presentation at the PolMeth XXXVI, Massachusetts Institute of Technology, Cambridge, MA July 18-20, 2019. A previous version was presented at the 2018 Annual Meeting of the Midwest Political Science Association, Chicago, IL, April 5–8, 2018. Work supported by NSF award SES 1523355. For assistance thanks to Mitchell Bosley, Preston Due, Kevin McAlister and Alejandro Pineda.

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Abstract

How to distinguish frauds from consequences of strategic behavior is the primary challenge for election forensics. Election forensics methods use empirical distributions of turnout and vote choices to identify fraudulent activity. Strategic behavior can affect these distributions. Generally, many types of election forensics methods interpret a multiplicity of modes in election data as indicators of frauds, but strategic behavior induces correlations among electors' behavior that can produce multimodalities. We present agent-based models (ABMs) designed to represent electors who decide how to act in elections based both on their own tastes and on their beliefs about what the other electors will do. In particular we define ABMs designed to match equilibria derived in models of wasted vote logic and of strategic abstention, and we construct ABMs that have simulated electors (agents) that use a combination of strategic abstention with wasted vote logic. To facilitate computing pivotal vote probabilities, our ABMs use a Poisson games formulation of the models instead of the fixed-electorate assumptions the formal models originally used. We replicate the formal models' basic equilibria. We apply election forensics statistical tools to data simulated using the ABMs. Second-digit tests and a likelihood finite mixture model are triggered: each signals "fraud" when in fact there is only strategic behavior. Such a result supports our idea to use ABMs to calibrate election forensics statistical methods so that fraud can be detected even in the presence of strategic buzz.

1 Introduction

A key challenge for election forensics—the field devoted to using statistical methods to determine whether the results of an election accurately reflect the intentions of the electors—is to be able to distinguish election results caused by election frauds from results produced by strategic behavior or other normal politics (Mebane 2013*a*, 2016). Here we report on our plan to use agent-based models (ABMs) (Axelrod 1997; Epstein 1999; Axelrod and Tesfatsion 2006; Bruch and Atwell 2015; de Marchi and Page 2014) to produce quantitatively specific realizations of theories of strategic election behavior and of election frauds. We plan ultimately to use ABMs to simulate various election systems at realistic scales, with agents that are involved in various kinds of networks and endowed with various distributions of attributes, where agents engage in equilibrium behavior such as is studied in formal models of rational election behavior (e.g. Palfrey and Rosenthal 1985; Cox 1994). Here we show one example applying election forensics statistical tools to vote counts simulated using an ABM that reflects electors acting strategically: the agents use a combination of strategic abstention with wasted vote logic. Eventually we will introduce various kinds of frauds into such systems. Among the questions for the long run will be to what extent can statistical methods for election forensics discriminate strategic actions from frauds, and can the methods help measure the extent and location of the frauds accurately.

1.1 Election Forensics

Election forensics studies counts of votes, counts of eligible voters and other traces of an election—preferably at low levels of aggregation such as tallies for each polling station—to produce evidence regarding what happened in the election. By starting with the numerical results and other measures of the voting process election forensics does not address the entirety of an election or address the full range of frauds that are possible (Lehoucq and Jiménez 2002; Lehoucq 2003; Magaloni 2006; Schedler 2006; Levitsky and Way 2010;

Minnite 2010; Birch 2011; Hyde and Marinov 2012; Svolik 2012; Wang 2012; Simpser 2013; Stokes, Dunning, Nazareno and Brusco 2013; Norris 2014). For example, if parties are excluded from the ballot, such an action may not produce distinctive patterns in the votes that are cast. But some violations of electoral integrity such as unfair access to campaign resources, wrongfully manipulated voter lists, vote buying, voter intimidation and other coercive actions may produce distinctive patterns in votes that statistics can detect. Ambiguities arise when such patterns might also be produced by strategic voting and other normal political activites such as forming coalitions.

Many methods for trying to detect election frauds have been proposed (e.g. Myagkov, Ordeshook and Shaikin 2009; Levin, Cohn, Ordeshook and Alvarez 2009; Shikano and Mack 2009; Mebane 2010*b*; Breunig and Goerres 2011; Pericchi and Torres 2011; Cantu and Saiegh 2011; Deckert, Myagkov and Ordeshook 2011; Beber and Scacco 2012; Hicken and Mebane 2015; Montgomery, Olivella, Potter and Crisp 2015; Mebane 2016; Rozenas 2017; Ferrari and Mebane 2017; Ferrari, McAlister and Mebane 2018). Methods based on the second significant digits of vote counts have been shown to respond both to normal political activities (strategic behavior, district imbalances, special mobilizations, coalitions) and to frauds (Mebane 2013*a*, 2014). Methods that examine the last digit of vote counts can be fooled if malefactors have sufficient control over the numbers (Mebane 2013*b*). All the methods can effectively identify various kinds of anomalies, but assessing whether the anomalies are due to frauds presents further challenges.

Some of the methods explicitly focus on the modality of election data. Some methods in this vein emphasize that unproblematic elections feature unimodal distributions of turnout and regular flows of votes—the latter are most compatible with assumed unimodal distributions for parties' shares of the votes (Myagkov, Ordeshook and Shaikin 2008, 2009; Levin et al. 2009). Other contributions connect "spiky" (hence multimodal) distributions of turnout and vote proportions to ideas about agents committing frauds in ways that they intend to be detected (Kalinin and Mebane 2011; Mebane 2013*b*; Rundlett and Svolik

2015). The sharpest contribution featuring multimodality is a model proposed by Klimek, Yegorov, Hanel and Thurner (2012) that stipulates a particular functional form according to which frauds occur.

Review of the theory (Borghesi 2009; Borghesi and Bouchaud 2010) that motivates the Klimek et al. (2012) conception suggests, however, that multimodal distributions may be as readily produced by strategic voting, coalitions and other strategic behavior as by election frauds that stem from malevolent activity (for details see Mebane (2016)). That theory, as it has been developed so far, does not imply that the distributions produced by strategic voting and by frauds are the same: currently the theory is not specifically quantitative. But the theoretical ambiguity about the origins of multimodality may carry over to make the parameters of the Klimek et al. (2012) conception ambiguous. Many of the other statistical methods also, on inspection, are crucially triggered by multimodalities and so may be subject to similar ambiguities. An election may appear to have frauds when in fact it has only robust politics featuring strategic activity by electors—by voters and would-be voters.

1.2 ABMs

We report on the initial steps of our plan to use ABMs to simulate strategic behavior and frauds in various election systems at realistic scales.[1] Previous efforts to use simulations motivated by strategic voting ideas to develop data to use to assess election forensics methods have not rigorously imposed equilibrium conditions, but have only roughly tried to approximate expected consequences of such behavior. For instance, simulations have shifted votes between candidates with references to the kind of behavior that happens when voters act according to wasted vote logic (e.g. Mebane 2013*a*). Perhaps such approaches capture some important qualitative aspects of what happens when voters act

[1] We recognize that some distinguish equation based models (EBMs) from ABMs (Parunak, Savit and Riolo 1998). EBMs begin with equations that capture relationships between observable characteristics (party preference, for instance), while ABMs begin with behaviors that define how individuals *interact* with one another (Parunak, Savit and Riolo 1998). The literature is divided on whether this differentiation makes sense; for any given computational model, there is a set of equations that could (theoretically) produce the same result (Bonabeau 2002; Epstein 1999). For the purposes of this paper, we do not differentiate.

that way, but there is no proof of that and there is little reason to believe the quantitative details about how many votes are affected are correct. While some formal models have equilibrium ideas with notions of frauds (e.g. Simpser 2013; Rundlett and Svolik 2015), no simulations have attempted to achieve quantitative precision about what happens.

Here we take only the first steps toward such a plan. We use ABMs to replicate equilibria obtained in two important formal models of elector behavior: the Palfrey and Rosenthal (1985) model of strategic abstention, as updated by Demichelis and Dhillon (2010) to include aspects of learning; and the Cox (1994) model of wasted vote behavior. In both instances we modify the models to use the conception of Poisson games (Myerson 1998, 2000). Going beyond replication, we also consider a model that combines strategic abstention with wasted vote behavior. Perhaps because of its complexity, such a combination has not previously been considered in published work. Others have studied election equilibria using the Poisson games formulation (e.g. Bouton and Gratton 2015). We take a Poisson game approach to make it feasible to compute pivotal voting probabilities (Myerson 1998).

Siegel (2018) discusses the relationship between computational models[2] and game theory, and in particular he considers how the steady states and limiting distributions of computational models are similar to the equilibrium concepts of game theory. He advocates a method to approximate the comparative statics of game theory: he proposes starting with a simple model then considering sequences of more complicated models. For our application we are not directly concerned with comparative statics, but we will ultimately care about identifying the domains of attraction for various "equilibrium" sets. We propose to generate simulated election data only from limiting points or sets. We use "equilibria" to refer to such sets, whether they be points, cycles or chaotic attractors.

[2]Siegel (2018, 746) prefers to speak in terms of computational models instead of ABMs, but his discussion covers the kinds of models we consider.

2 Models

We start by developing models that resemble, respectively, the Palfrey and Rosenthal (1983, 1985) model of strategic abstention and the Cox (1994) model of wasted vote behavior. For the former model we adopt the "learning" approach previously used by Demichelis and Dhillon (2010) to make points about the number of equilibria in some variants of the Palfrey and Rosenthal (1983, 1985) model. For the Cox (1994) model we use an iterative setup analogous to the framing by Fey (1997) that involves polling reports of out-of-equilibrium results.

All the formal models refer to electors (for Palfrey and Rosenthal 1985) or voters (for Cox 1994) formulating strategies based on pivotal vote calculations. Such calculations involve each elector or voter calculating the probability that its vote creates or breaks a tie for winning status—winning office or winning a seat. The models were originally developed with the assumption that the size of the electorate or the number of voters is fixed and known, and these models use multinomial formulations to describe the probability of vote counts and therefore pivotal probabilities. Such multinomial tie probabilities are arduous to compute when the number of electors or voters is large, so we adopt the idea that the number is known only to be drawn from a Poisson distribution (Myerson 1998, 2000) which simplifies computing pivotal probabilities. Using the Poisson assumption and Skellam probabilities (Skellam 1946) we can replicate the most important features of the Palfrey and Rosenthal (1985) and Cox (1994) models, but we can also explore versions of these models that are infeasible to investigate using paper-and-pencil methods. We consider a "learning-with-polling" model that combines strategic abstention with wasted vote behavior.

2.1 Strategic Abstention

Palfrey and Rosenthal (1983, 1985) consider the classic instrumental formulation for the net benefits R from voting in an election with two alternatives:

$$R = pB - C + D\,, \tag{1}$$

where p is the probability that an individual's vote decides the election outcome (is "pivotal"), B is the difference in benefits for the individual if the individual's more preferred choice wins instead of the alternative, C is the cost of voting and D is the direct benefit from voting. Neither C nor D depend on the election outcome but depend only on the individual deciding to participate in the election by voting. Palfrey and Rosenthal (1985) demonstrate the existence of equilibria with positive turnout given a variety of assumptions about the distributions of costs and uncertainty. Palfrey and Rosenthal (1985) argue that turnout, when positive, is usually low.

Demichelis and Dhillon (2010) consider symmetric equilibria given the same net benefits and structure as Palfrey and Rosenthal (1985), while adding learning dynamics. Demichelis and Dhillon (2010) normalize (1) by setting $B = 1$, in which case a necesssary condition for an individual to vote is $p - 2c \geq 0$, where $c = C - D$. Letting q denote the symmetric mixed strategy for all players and using $g(q, M)$ to denote the probability of being pivotal, Demichelis and Dhillon (2010, 881) suppose that q changes over time according to

$$\frac{dq}{dt} = K(q) \tag{2}$$

where sign $K(q) = \text{sign}(g(q, M) - 2c)$.[3] Demichelis and Dhillon (2010) investigate the dynamic stability of equilibria.

2.2 Wasted Vote Logic

Cox (1994) considers situations where each voter casts a single vote, M alternatives "win"

[3] Demichelis and Dhillon (2010, 881) note that, "If the adjustment steps are small enough, taking a discrete adjustment process would give essentially the same results." Our ABMs use such discrete adjustments.

the election by gaining seats and the M alternatives that have the M highest numbers of votes win the seats. Assuming that each voter maximizes expected utility based on pivot probabilities, that utilities are diverse,[4] that the distribution of preferences is common knowledge and that expectations about the election outcome are publicly generated (to the point that Cox (1994, 610) imposes a rational expectations condition), Cox (1994) shows that in the limit, as the number of voters grows, there are two kinds of equilibria. With "Duvergerian" equilibria only the first loser (in $M+1$-th place) receives a nonnegligible proportion of the votes while other losers get nothing, and with "non-Duvergerian" equilibria the losing alternatives have either the same positive proportion of votes as does the first loser or zero votes. Cox (1994) notes that the formal findings about exact ties between alternatives depend on particular simplifying features of the model. The empirical tests he conducts of the theory are not quite so sharp.

2.3 Pivotal Probabilities

A key quantity in the considered models is the probability of an individual voter casting a pivotal vote. Demichelis and Dhillon (2010) and Cox (1994) assume that electorate sizes are fixed and commonly known. This allows calculation of pivotal probabilities through exact combinatorial computation or multinomial assumptions. Though this approach can yield results, calculation of these probabilities is computationally demanding. In order to derive computationally efficient pivotal probabilities, we rely on an assumption that the electorate size is only known up to the distribution. In particular, we apply the large Poisson games framework derived by Myerson (2000).

There are N_e electors, but elector $i \in \{1, \ldots, N_e\}$ has incomplete knowledge about the number of electors. Elector i assumes N_e is a Poisson distributed random variable with

[4] Cox (1994, 610) assumes that the electorate includes voters with all possible preference orders.

mean μ:

$$N_e \sim Pois(\mu)\,. \tag{3}$$

Elector i also receives information about the proportions of voters in the electorate that will take certain actions: for candidate $h \in \{1, \ldots, H\}$, it learns the proportion p_h of voters that are going to cast a vote for h. Let n_h be the number of voters that vote for candidate h. Given p_h and the properties of Poisson random variables, n_h is also treated as a Poisson random variable:

$$n_h \sim Pois(p_h \mu)\,.. \tag{4}$$

To compute pivotal probabilities we are not concerned with the raw vote counts, rather we want to know the probability distribution of one candidate's vote counts relative to other candidates. Let X and Y be independent Poisson random variables. Then

$$P(X - Y = z) = \mathcal{S}(z; \mu_x, \mu_y) = \exp(-(\mu_x + \mu_y)) \left(\frac{\mu_x}{\mu_y}\right)^{\frac{z}{2}} I_z\left(2\sqrt{\mu_x \mu_y}\right) \tag{5}$$

where $I_z(\cdot)$ is a Bessel function of the first kind. $\mathcal{S}(\cdot)$ is the Skellam distribution (Skellam 1946). Assuming Poisson vote counts, there are two situations in which i's vote for party h is pivotal: its vote pushes party h out of a tie and into a winning position or its vote pushes party h into a tie for a winning position. Each elector evaluates pivotal probabilities based on an imagined electorate that does not include its own vote. Using $\tilde{\mu}_h = p_h(\mu - 1)$, for a single winner election where the candidate with the most votes wins, this implies that an elector's probability of casting a pivotal vote for party h (pv_h) is:

$$pv_h = \sum_{j \neq h} [P(\tilde{\mu}_h - \tilde{\mu}_j = 0 | \tilde{\mu}_h, \tilde{\mu}_j \geq \tilde{\mu}_m \; \forall \; m \neq j, h) \\ + P(\tilde{\mu}_h - \tilde{\mu}_j = -1 | \tilde{\mu}_h, \tilde{\mu}_j \geq \tilde{\mu}_m \; \forall \; m \neq j, h)] \prod_{m \neq j,h} P(\tilde{\mu}_h - \tilde{\mu}_m > 0) \, . \tag{6}$$

In words, the probability that it casts a pivotal vote is the probability that it breaks or creates a tie times the probability that party h is at least in second place.[5] We can define pv_h in terms of Skellam distributions:

$$pv_h = \sum_{j \neq h} [\mathcal{S}(0; \tilde{\mu}_h, \tilde{\mu}_j) + \mathcal{S}(-1; \tilde{\mu}_h, \tilde{\mu}_j)] \prod_{m \neq j,h} \left(\sum_{w=1}^{\infty} \mathcal{S}(w; \tilde{\mu}_h, \tilde{\mu}_m) \right) . \tag{7}$$

This value can be evaluated computationally using Skellam implementations in various statistical libraries.

2.4 Strategic Abstention with Wasted Vote Logic

To combine strategic abstention with wasted vote logic we let there be several parties—not merely two parties—competing for one seat. The first step is to formulate the pivotal probabilities. Then "learning" and "polling" proceed as previously.

Notation and definitions here follow, as closely as possible, those from Demichelis and Dhillon (2010). Let $i = 1, \ldots, N_e$ index electors, and let $j, h = 1, \ldots, J$ index parties. n_h as defined in (4) is the number of voters that vote for candidate $h = 1, \ldots, J$. Let ℓ_{jh} denote the probability that j and h are the top two parties, with $\ell_{jj} = 0$. Given that j and h are the top two parties, define the following conditional probabilities:

- ϕ^i_{1jh}: conditional probability that elector i voting for j would break a tie between parties j and h;

[5] Three-way ties are ignored.

- ϕ^i_{2jh}: conditional probability that elector i voting for j would create a tie between parties j and h;

- ϕ^i_{3jh}: conditional probability that elector i is not pivotal in relation to parties j and h (does not break or make a tie between them) and party j leads;

- ϕ^i_{4jh}: conditional probability that elector i is not pivotal in relation to parties j and h (does not break or make a tie between them) and party h leads.

Define $\alpha^i_{ajh} = \phi^i_{ajh}\ell_{jh}$, $a = 1, \ldots, 4$. Let u^i_j be the utility (or benefit) to elector i if party j wins. Using E^i_{jh} and $E^i_{\varnothing jh}$ to denote respectively the expected utility, exclusive of the net costs of the act of voting, for elector i from voting for j over h and from abstaining instead of voting for j over h, assuming ties are broken at random we specify

$$E^i_{jh} = \alpha^i_{1jh}u^i_j + \alpha^i_{2jh}\frac{u^i_j + u^i_h}{2} + \alpha^i_{3jh}u^i_j + \alpha^i_{4jh}u^i_h \tag{8a}$$

$$E^i_{\varnothing jh} = \alpha^i_{1jh}\frac{u^i_j + u^i_h}{2} + \alpha^i_{2jh}u^i_h + \alpha^i_{3jh}u^i_j + \alpha^i_{4jh}u^i_h\,. \tag{8b}$$

Hence, including the net cost of voting c^i, the expected net benefit to i from voting for j is

$$R^i_j = \sum_{h\neq j}[E^i_{jh} - E^i_{\varnothing jh}] - c^i \tag{9a}$$

$$= \sum_{h\neq j}(\alpha^i_{1jh} + \alpha^i_{2jh})(u^i_j - u^i_h) - c^i \tag{9b}$$

$$= \sum_{h}\mathcal{P}^i_{jh}B^i_{jh} - c^i\,, \quad \mathcal{P}^i_{jh} = \alpha^i_{1jh} + \alpha^i_{2jh}\,, \quad B^i_{jh} = u^i_j - u^i_h\,. \tag{9c}$$

(cf. McKelvey and Ordeshook 1972, 41–42). Given the Poisson assumptions $\mathcal{P}^i_{jh}$ can be evaluated using Skellam probabilities:

$$\mathcal{P}^i_{jh} = [\mathcal{S}(0; \tilde{\mu}_j, \tilde{\mu}_h) + \mathcal{S}(-1; \tilde{\mu}_j, \tilde{\mu}_h)] \prod_{m\neq j,h}\left(\sum_{w=1}^{\infty}\mathcal{S}(w; \tilde{\mu}_h, \tilde{\mu}_m)\right)\,. \tag{10}$$

Evidently if every elector has the same belief about the vote proportions p_j, then $\mathcal{P}^i_{jh}$ is the same for all electors: $\mathcal{P}^i_{jh} = \mathcal{P}_{jh}$, $i = 1, \ldots, N_e$. Such is the case given an assumption that information about the current (not necessarily in equilibrium) vote counts are conveyed to everyone via a "poll" (i.e., "publicly generated"). In this case

$$R^i_j = \sum_h \mathcal{P}_{jh} B^i_{jh} - c^i \,. \tag{11}$$

We consider models in which electors' strategies are composed of a mixed strategy for abstaining and a conditional pure strategy for party choice. Let $q^i \in [0, 1]$ denote the mixed strategy for elector i: q^i is the probability that elector i votes. Given $\mathcal{P}^i_{jh}$, if an elector votes its choice is

$$\begin{aligned} j &= \operatorname*{argmax}_j R^i_j && \text{(12a)} \\ &= \operatorname*{argmax}_j \sum_h \mathcal{P}^i_{jh} B^i_{j,h} \,. && \text{(12b)} \end{aligned}$$

Given q^i and $\mathcal{P}^i_{jh}$, $i = 1, \ldots, N_e$, using the indicator function $\mathcal{I}(\cdot)$ the number of votes for j is

$$\begin{aligned} n_j &= \sum_i q^i \mathcal{I}\left(j = \operatorname*{argmax}_j \sum_h \mathcal{P}^i_{jh} B^i_{j,h} \right) \\ &= \sum_i q^i \mathcal{I}^i_j \,, \quad \mathcal{I}^i_j = \mathcal{I}\left(j = \operatorname*{argmax}_j \sum_h \mathcal{P}^i_{jh} B^i_{j,h} \right) \,. \end{aligned}$$

A learning formulation for the mixed strategies, akin to (2) from Demichelis and Dhillon (2010, 881), is

$$\begin{aligned} \frac{dq^i}{dt} &= K \left(\sum_j \mathcal{I}^i_j \sum_h \mathcal{P}^i_{jh} B^i_{j,h} - c^i \right) && \text{(13a)} \\ &= K \left(\max_j \sum_h \mathcal{P}^i_{jh} B^i_{j,h} - c^i \right) \,. && \text{(13b)} \end{aligned}$$

Notice that (2) is for a symmetric mixed strategy q that every elector uses, while (13b) is formulated in terms of individualized mixed strategies that are diverse across electors.[6] We can impose the restriction that for all electors $q^i = q$.[7] Diverse q^i are feasible to investigate when using ABMs but not generally when relying on paper-and-pencil analytic methods. In connection with ABMs we use a difference equation form of the q^i update: for some small value $k^{i,t}$,

$$q^{i,t+1} = q^{i,t} + \text{sign}\left(\max_j \sum_h \mathcal{P}^{i,t}_{jh} B^i_{jh} - c^i\right) k^{i,t}\,, \tag{14}$$

where the $\mathcal{P}^{i,t}_{jh}$ are evaluated using $p^t_j = n^t_j/N_e$, $j = 1, \ldots, J$.[8]

If all electors have positive net costs, i.e., if $c^i > 0$ for all $i = 1, \ldots, N_e$, then the dynamics of (13b) or (14) can produce situations even for small N_e where all $q^i = 0$ hence $n_j = 0$ for all parties—no one votes! One can avoid such a situation by ensuring that $c^i < 0$ for some electors. Or one can introduce a jump in the update rules: if $n_j = 0$ for all j, then set all $\mathcal{P}^i_{jh} = 1$ (or $\mathcal{P}^{i,t}_{jh} = 1$) and set $q^i = 1$ (or $q^{i,t+1} = 1$) for all i that have $u^i_j > c^i$ for some j. Then n_j (or n^{t+1}_j) reflects every elector acting sincerely. The system then continues.

[6] If there are two parties and electors' utilities (u^i_1, u^i_2) are either $(1,0)$ or $(0,1)$, then (13b) defines a version of (2) for individualized mixed strategies with utilities as in Demichelis and Dhillon (2010).

[7] The trick when defining a learning dynamic if $q^i = q$ for all electors is to decide what information each elector has about other electors' utilities and costs. If $\mathcal{U}$ is a distribution of utilities, $\mathcal{B}$ is the domain of B^i_{jh} values, and $C(q)$ (from Demichelis and Dhillon 2010) is a distribution of net costs expressed as a function of the mixed strategy, then in line with Demichelis and Dhillon (2010, 888) we might specify

$$\frac{dq}{dt} = K\left(\sum_j \int_{\mathcal{B}} \mathcal{I}_j \sum_h \mathcal{P}_{jh} B_{jh} d\mathcal{U} - C(q)\right)\,.$$

[8] In practice we define $k^{i,t}$ in terms of $q^{i,t}$ to ensure $k^{i,t}$ is small enough not to produce artificial boundary violations:

$$k^{i,t} = 10^{\lfloor \log_{10}(\min(1-q^{i,t}, q^{i,t}))\rfloor - 2} \;:\; k^{i,t} \geq 10^{-6}\,.$$

E.g., if $q^{i,t} = .05$ then $k^{i,t} = .0001$.

3 Replications

We present examples of ABMs built on a Poisson games foundation designed to replicate basic results from Demichelis and Dhillon (2010) and Cox (1994). For Demichelis and Dhillon (2010) our goal is to replicate Figure 2 in their paper. For Demichelis and Dhillon (2010) we stick as closely as we can to the assumptions they used. Cox (1994) does not provide an analogous specific replication target, so we illustrate how ABMs can produce Duvergerian equilibria.[9] Even though the scaling of utilities is inconsequential for our ABM, we restrict utilities to be in the unit interval, as Cox (1994) assumes.

3.1 Strategic Abstention

We start by focusing on scenarios in which the two candidates A and B are supported by the same proportions $s_A = 1 - s_B = .5$ of electors, electors have the same net costs of voting $c_i = c$, $i = 1, \ldots, N_e$, and all electors have the same mixed strategy probability $q_i = q$ of voting. More specifically, we initialize each run of the model where half of the electors support candidate A and the other half support candidate B, so $N_A = N_B = N_e/2$; and we also define a value for q and c s.t. $q \sim U(0,1)$ and $c \sim U(0, 0.5)$. We use an expected electorate size $\mu = 400$.

Let n_A and n_B be, respectively, the expected numbers of supporters of candidates A and B that would go vote if the election happened at the current iteration of the model. Then

$$n_j = \sum_{i}^{N_j} q^i \tag{15}$$

where $i = 1, \ldots, N_j$ are the electors that support candidate $j \in \{A, B\}$.

At each iteration t, each elector i updates its probability q^i of voting, according to the

[9] To produce non-Duvergerian equilibria requires tuning utilities in ways we have not yet pursued.

following difference equation:

$$q^{i,t+1} = q^{i,t} + K_i(\tilde{\mu}_A^t, \tilde{\mu}_B^t) \tag{16}$$

To implement $K(.)$ (recall (2)) we follow Demichelis and Dhillon (2010) and use

$$K_i(\tilde{\mu}_A, \tilde{\mu}_B) = k \ \text{sign}(g_i(\tilde{\mu}_A, \tilde{\mu}_B) - 2c) \tag{17}$$

where $k > 0$ represents the learning speed mentioned in Demichelis and Dhillon (2010)[10], and $g_i(\tilde{\mu}_A, \tilde{\mu}_B)$ is the function that calculates the probabilities of elector i being to create or break a tie in the election:

$$g_i(\tilde{\mu}_A, \tilde{\mu}_B) = \begin{cases} \mathcal{S}(0, \tilde{\mu}_A, \tilde{\mu}_B) + \mathcal{S}(-1, \tilde{\mu}_A, \tilde{\mu}_B) & \text{if } i \text{ supports A} \\ \mathcal{S}(0, \tilde{\mu}_B, \tilde{\mu}_A) + \mathcal{S}(-1, \tilde{\mu}_B, \tilde{\mu}_A) & \text{if } i \text{ supports B} \end{cases} \tag{18}$$

where $\mathcal{S}$ is defined in (5). $\mathcal{S}(0, ...)$ and $\mathcal{S}(-1, ...)$ represent, respectively, the probability of elector i being able to untie the election and the probability of i being able to tie the election.[11] With these specifications, (16) is a difference equation implementation of the dynamic described in note 7 with only two parties ($J = 2$) as described in note 6.

The stability analysis summarized by Figure 2 in Demichelis and Dhillon (2010) suggests that equilibrium points reached by (16) should depend not only on c but also on the initial value specified for $q^t = q^{t_0}$.[12] This is because Demichelis and Dhillon (2010)

[10] We set k adaptively as described in note 8.

[11] Given that the support of the Skellam density does not include zero, we set $\mathcal{S} = 0$ if $\tilde{\mu}_A = 0$ or $\tilde{\mu}_B = 0$.

[12] Observed convergence in this model has been of two types, which we call hard convergence and bouncing, hence we needed two convergence criteria to stop the simulation. A given run was considered to hard-converge if the values of n_A and n_B remained the same for 50 iterations. Otherwise, a run could be stopped due to achieving stable bouncing if n_A and n_B kept alternating between two exact same values every two iterations for 50 iterations. Such bouncing is due to our specifying a lower bound for each $k^{i,t}$: specifically we set $k^{i,t} \geq k_{\min} = .000001$; using smaller $k_{\min}$ values substantially increased the number of iterations while only slightly reducing the frequency of bouncing. That includes allowing $k_{\min}$ be as small as floating point machine precision. In both cases, since we are operating with floating point arithmetic, we consider equality over iterations as being equality up to $\epsilon = 10^{-6}$; using smaller ϵ values does not materially change any results.

identify three equilibria whenever $c > c_{\min}$,[13] but only "low turnout" and "full turnout" equilibria are dynamically stable. The suggestion is that there is a nonempty domain of initial values q^{t_0} that produce the "full turnout" equilibrium.

In fact using our Poisson electorate assumption we find equilibrium depends only on c. Figure 1 shows the final probability q of voting versus the net cost of voting c. We show plots over a grid of c values separately for different values of initial q^{t_0}. All electors are expected to vote ($q = 1$) when $c = 0$ and indeed when $c < \tilde{c}_{\min}$: $\tilde{c}_{\min}$ is the Poisson electorate variant of what is $c_{\min}$ with fixed electorate assumptions. Whenever $c > \tilde{c}_{\min}$ we find equilibrium $q > 0$, with equilibrium q becoming very small as c rises. Because of the way utilities are scaled in our model (we use (14) to implement this replication) equilibrium q is positive for c as large as $c = 1$ instead of the upper bound of $c = .5$ that applies for Demichelis and Dhillon (2010). The point about equilibrium depending only on c is that we obtain the same equilibrium q regardless of the starting value. Even when initial $q^{t_0} = 1$, equilibrium q has the same value it does with as any other initial value.

*** Figure 1 about here ***

Our failure to reproduce a dynamically stable "full turnout" equilibrium when $c > \tilde{c}_{\min}$ is not surprising. Using fixed-electorate assumptions, Palfrey and Rosenthal (1985, 71–73) prove that the wide range of full turnout equilibria does not exist for electorates as large as we are using. Indeed, whenever n_A or n_B is not small, the Skellam distribution function returns small values so that $g(.) < 2 \cdot c$, which forces q to decrease.

Figure 1 also demonstrates that the variation in the realized electorate size N_e that the Poisson electorate assumption introduces makes no essential difference for the pattern of equilibrium q values. Figure 1(a) shows results when $N_e \sim Pois(400)$ is independently drawn for every run of the simulation, while Figure 1(b) shows results when we use exactly $N_e = 368$ (the same randomly drawn value) for every run. There is no perceptible

[13] Using equation (4) of Demichelis and Dhillon (2010, 877) for an electorate fixed at size $M = 400$, $c_{\min} = .02821$.

difference in outcomes due to randomly varying the electorate size N_e.

Except for the differences regarding the "full turnout" equilibrium when $c > \tilde{c}_{\min}$, the results shown in Figure 1 replicate the main equilibrium results shown in Figure 2 of Demichelis and Dhillon (2010). In particular the stable "low turnout" equilibrium locus is reproduced.

3.2 Wasted Vote Logic

Our replication of the Cox (1994) wasted vote model functions as follows. First we specify a number of candidates and a number of electors. For each elector $i = 1, \ldots, N_e$ we generate one utility value $u_j^i \in [0, 1]$ per candidate $j = 1, \ldots, J$, with $\min_j u_j^i = 0$ and $\max_j u_j^i = 1$. Each utility is generated using $\underline{u}_j^i \sim \text{Unif}(0, 1)$, then the $\underline{u}_j^i$ values are rescaled for each elector to produce u_j^i. Each elector has a utility value for every candidate. These utilities are assumed to be von Neumann-Morgenstern consistent and indicative of at least a weak preference ordering over the candidates.[14] To match Cox (1994), we ensure that across electors every possible rank order of candidates is present in the utility vectors (the exact proportion that has each rank is determined randomly). Voters then start by choosing any candidate other than their least preferred candidate to be their initial choice. We let voters "state" any nondominated[15] initial choice to capture effects of variation in the initial beliefs that we have not yet modeled explicitly. Once they make this initial choice, they use public information about the distribution of all voters' current choices to make a strategically optimal choice, v^i, according to

$$\begin{aligned} u_h^{i*} &= \sum_{j=1}^{J} T_{h,j}(u_h^i - u_j^i) \\ v^i &= \underset{h=1,\ldots,J}{\text{argmax}}[u_h^{i*}] \end{aligned} \tag{19}$$

[14]Given the double-precision pseudorandom numbers we use to generate utilities, no two utility values in $(0, 1)$ should be the same.

[15]An elector never votes for its least preferred candidate.

where $T_{h,j}$ is the probability that an elector casts a pivotal vote for h against j. Using (7), we set $T_{h,j} = pv_h$ with n_h being the current "stated" vote counts for candidates $h, j = 1, \ldots, J$.[16]

This procedure is iterative: each elector simultaneously updates its choice to be strategically optimal conditional upon a) its utilities and b) every other electors' current choice. But after each step the state of the world has changed, so the strategic decisions that were conditioned on other electors' previous choices are no longer strategically optimal. So in the next iteration of the model every elector simultaneously updates its choices given each other electors' current choices. The electors continue to update in this manner until no electors' strategic choices change for an arbitrary user-selected number of iterations. At this point electors' choices are in a strategic equilibrium.

Between any two runs of the model, there are therefore two elements of randomness. The central source of randomness is the distribution by which electors' utilities are initialized. The second source of randomness is the arbitrary selection of voters' initial candidate choice. Different results can be obtained by changing the distribution of electors' utilities, or by holding those utilities fixed while rerunning the model with different initial choices. The first option is a completely different initial setup, which generates electors that have meaningfully different preference structures. The second option is a perturbation of the same initial electors and candidates, changing only the electors' initial selection. A run of the model with the initial selection varied yields an equilibrium in the set of equilibria that are obtained by perturbing a model slightly away from the initial setup. Such differences reflect the existence, generally, of multiple equilibria for any given set of utilities (Myerson and Weber 1993).[17]

With these specifications of the model, our replication so far yields only one of the two different outcomes which were derived by Cox (1994): over more than 15,000 runs we find only the 2-candidate or $M+1$ "Duvergerian" result in which one candidate wins, another

[16] In fact we use use (14) to implement this replication: all net costs are negative, so all $q^i = 1$, and $J > 2$.

[17] We do not attempt to identify all equilibria nor to characterize the domains of attraction for the equilibria.

candidate gets second place, and every other candidate falls to zero or near-zero votes. Using uniform independent utilities evidently makes it unlikely that configurations of utilities occur that are conducive to the existence of non-Duvergerian equilibria, in which one candidate wins and all of the other candidates tie for last. We find the same kinds of results when we run the model with several utility-generating distributions.[18] We conjecture that by tuning the utilities the model can produce non-Duvergerian equilibria, but we have not yet attempted such tuning. We can produce different equilibria given the same utilities by perturbing the initial choice of favored candidate (cf. Section 4.1.2).

4 Extensions

We consider extensions beyond the preceding replication-minded examples. Some extensions we execute. These are simple generalizations of the strategic abstention and wasted vote logic models done using the combined model of section 2.4. Other extensions describe a couple of ways we might extend the combined model to include a wider array of strategic features in the future.

4.1 Strategic Abstention with Wasted Vote Logic

We produce simulations using (14) for four purposes. First we illustrate generalizations of Cox (1994) and Demichelis and Dhillon (2010) by allowing wasted vote logic and strategic abstention simultaneously with individualized abstention mixed strategies. Second we show that multiple equilibria can occur with substantial turnout in a moderate-sized electorate. For this multiple equilibria example we illustrate the trajectories (or orbits) that can occur—from initial values to equilibrium—both for aggregate vote counts and for individual electors. Finally we use (14) to generate sets of vote counts to which we apply election forensics tools. The election forensics question is whether even the simple kind of

[18]In addition to the uniform distribution, distributions we tried include standard normal, power law (exponent 2), logistic ($\mu = 2, s = 1$) and Beta ($\alpha = 0.01, \beta = 0.01$) distributions.

strategic behavior represented by the model of strategic abstention with wasted vote logic can trigger "frauds" signals.

4.1.1 Generalizations

Unlike Demichelis and Dhillon (2010) we allow individualized mixed strategies q^i instead of the single symmetric mixed strategy that is their principal focus. For half of the simulations we make all net costs positive with $c^i \sim \text{Beta}(10, 10)$, $c^i \sim \text{Beta}(.5, .5)$ or $c^i \sim \text{Beta}(1, 1) = \text{Unif}(0, 1)$. For the other half of the simulations we create c^i the same way but then multiply about 1/4 of the c^i values (randomly selected) in each electorate by -1.[19] We let the initial strategy values be uniformly distributed: $q^{i,t_0} \sim \text{Unif(0,1)}$.[20] We specify the number of parties randomly from $J \in \{2, 3, 4, 5, 6\}$. For each elector $i = 1, \ldots, N_e$ we generate utility values $u^i_j \in [0, 1]$ with $\min_j u^i_j = 0$ and $\max_j u^i_j = 1$. Each utility is generated using $\underline{u}^i_j \sim \text{Beta}(\alpha_1, \alpha_2)$ with $\alpha_1 \sim \text{Unif(.1,5)}$ and $\alpha_2 \sim \text{Unif(.1,5)}$, then the $\underline{u}^i_j$ values are rescaled for each elector to produce u^i_j. We performed 2000 simulation runs.

In all simulation runs either two parties finish with positive expected vote counts while all other parties (when $J > 2$) have expected counts of zero, or all parties have expected vote counts of zero. All of the instances where all expected vote counts are zero occur when all net costs are positive and generated by $c^i \sim \text{Beta}(10, 10)$. Figure 2(a) shows a scatterplot of the mean costs ($\bar{c} = N^1_e \sum c^i$) and mean equilibrium mixed strategies ($\bar{q} = N^1_e \sum q^i$) when $c^i \sim \text{Beta}(10, 10)$: for 97 of the 333 runs, $\bar{q} = 0$ which implies that all $q^i = 0$, so no one votes and all expected vote counts are zero.[21] In Figure 2(d), which shows $\bar{c}$ and $\bar{q}$ computed using only electors with positive net costs, many $\bar{q}$ values are very small but only one of those $\bar{q}$ values equals zero. None of the small-looking $\bar{q}$ values in the other

[19] We use $\text{Unif}(0, 1) < .25$ to decide whether to multiply c^i by -1.

[20] Demichelis and Dhillon (2010) say their results hold for mildly dispersed initial q^i values. Plainly $q^{i,t_0} \sim \text{Unif(0,1)}$ differs substantially from "mildly dispersed."

[21] For the all-zero instances in Figure 2(a) we might have adopted the "jump" approach described on page 12, but we found that while sometimes a run went to a non-zero solution after starting over with everyone voting sincerely for their most preferred party, sometimes the run went right back to the all-zero solution. Instead of adopting some arbitrary rule to deal with such cyclical trajectories, we decided to leave Figure 2(a) as it is and add simulations that include some electors with negative costs. Hence Figures 2(d–f).

scatterplots are zero, either.

The average mixed strategy $\bar{q}$ for electors with positive net costs varies considerably over simulations. In each case the value of $\bar{q}$ traces to the proportion of such electors that have equilibrium $q^i = 1$, which relates to the proportion of electors that have smaller c^i values. With the U-shaped distribution of $c^i \sim \text{Beta}(.5, .5)$ typically the highest proportion of electors have small c^i values, with $c^i \sim \text{Beta}(1, 1)$ typically the proportion of such electors is second highest and with the inverted-U-shaped distribution of $c^i \sim \text{Beta}(10, 10)$ the proportion of such electors is typically least. $\bar{q}$ is typically greater in Figures 2(a–c) than in the corresponding Figures 2(d–f) because with about a quarter of electors having $c^i < 0$ and consquently $q^i = 1$, all such electors vote which tends to reduce the pivotal probability values that affect q^i for electors that have $c^i > 0$.

*** Figure 2 about here ***

When net costs are rescaled to extend over smaller values, the resulting distributions of equilibrium $\bar{q}$ values traces a declining pattern as $\bar{c}$ increases similar to that observed for the symmetric mixed strategy q in Figure 1. Figure 3 displays this result. We rescale net costs c^i according to $c^i = s\underline{c}^i$ for $s \in \{.05, .1, .2, .5, 1\}$ where $\underline{c}^i$ is randomly generated according to Beta(10, 10), Beta(.5, .5) or Beta(1, 1). As the rescaling factor is smaller so that c^i is smaller, equilibrium $\bar{q}$ tends to be larger. The variance of the $\bar{q}$ values varies over s partly because the variance of c^i decreases with s^2.

*** Figure 3 about here ***

Figure 4, which shows scatterplots of net costs and equilibrium mixed strategies for individual electors taken from single simulations of (14), illustrates what we mean when we say that $\bar{q}$ traces to the proportion of such electors that have equilibrium $q^i = 1$. Very few electors have equilibrium mixed strategies that are not either zero or one. The exact distribution of the number of electors in each run from Figure 2 that have equilibrium

$q^i \in (0,1)$ is reported in Table 1. For the Figure 2 runs that have all net costs positive the modal number of electors with equilibrium $q^i \in (0,1)$ is one for $c^i \sim \text{Beta}(.5,.5)$ and $c^i \sim \text{Beta}(1,1)$ and zero for $c^i \sim \text{Beta}(10,10)$. For the Figure 2 runs that have about a quarter of electors with negative net costs the modal number of electors with equilibrium $q^i \in (0,1)$ is zero. The largest number of electors we observe with equilibrium $q^i \in (0,1)$ is eleven in one run with $c^i \sim \text{Beta}(.5,.5)$. Moreover as Figure 4 suggests, many of the values $q^i \in (0,1)$ are much closer to zero or one than they are to .5. Table 2 shows that about one-third of the Figure 2 runs with all net costs positive have one or two $q^i \in [.25,.75]$, but when about a quarter of electors have negative net costs only about one-sixth of runs with $c^i \sim \text{Beta}(.5,.5)$ or $c^i \sim \text{Beta}(1,1)$ have one elector with such a mixed strategy and none of the runs with $c^i \sim \text{Beta}(10,10)$ have such an elector.

*** Figure 4 and Tables 1 and 2 about here ***

4.1.2 Multiple Equilibria

As Myerson and Weber (1993) demonstrate for voting systems generally, multiple equilibria exist in the model of strategic abstention with wasted votes. We show some of the different equilibria that can arise with identical voter utilities.

Figure 5 shows scatterplots of net costs and equilibrium mixed strategies for individual electors taken from single simulations of (14) with $J = 5$ parties, electorate size $N_e = 1017$ ($\mu = 1000$) and costs compressed into $c^i \in (.015,.04)$ by using $c^i = .015 + .025\underline{c}^i$ for $\underline{c}^i \sim \text{Beta}(1,1)$. All four runs use identical net costs and identical utilities generated from Beta(1,1). Diverse initial values come from manipulating q^{i,t_0}: Figure 5(a) begins with $q^{i,t_0} \sim$ Unif(0,1); Figure 5(b) begins with $q^{i,t_0} = 0.01$ for all electors; Figure 5(c) begins with $q^{i,t_0} = 0.99$ if elector i most prefers party $j \in \{3,4\}$ else $q^{i,t_0} = 0.1$; and Figure 5(d) begins with $q^{i,t_0} = 0.99$ if elector i most prefers party $j \in \{2,3\}$ else $q^{i,t_0} = 0.1$.

*** Figure 5 about here ***

In each scenario shown in Figure 5 two different parties have positive expected vote counts in equilibrium while other parties have zero votes. In (a) parties 1 and 5 have $n_j > 0$ and in (b) parties 1 and 4 do. In both cases n_1 is biggest. In (c) parties 3 and 4 have $n_j > 0$ ($n_4 > n_3$), and in (d) parties 2 and 3 have $n_j > 0$ ($n_3 > n_2$). Notably the last equilibrium (d) occurs even though parties 2 and 3 are the parties whom the least electors most prefer: 160 and 153 electors, respectively, most prefer parties 2 and 3, while 251, 225 and 228 electors respectively most prefer parties 1, 4 and 5. In scenario (d), the least most-preferred party wins. Remarkably also scenario (d)—that has the, in a sense, least liked parties winning and finishing second—is the scenario of the four that has the highest expected turnout. In (d) each party with a positive number of votes receives more expected votes than does any party in the other three scenarios. These are only a haphazardly selected few of the distinctive equilibrium outcomes that exist given the stipulated utilities and net costs, and only a few of the initial conditions that produce them.

4.1.3 Trajectories

Even though when simulating electorates for election forensics we are interested only in equilibria, it is interesting at least for the purpose of better understanding dynamics to examine the trajectories followed by aggregates such as vote counts as well as paths followed by individual agents or groups of agents. In addition, given that we are simulating elections, we must consider that sometimes the equilibrium may be a cycle of several points, and at the extreme we may encounter equilibrium chaotic dynamics that while confined to a bounded subspace is not strictly speaking cyclical.

In our current model using (14) we encounter "bouncing" limiting states that trace to our imposing a positive lower bound for $k^{i,t}$ (recall note 8) and that we treat as equilibria. In "bouncing" there are a few electors—sometimes only one elector—that oscillate between pairs of values $q^{i,t}$ and $q^{i,t+1}$ that differ very little from one another.[22] We also sometimes

[22]Usually differences are the order of 10^{-6}, which matches the minimum value we set for $k^{i,t}$. As mentioned in note 12, when we use smaller minimum values for $k^{i,t}$ some but not all instances of bouncing are eliminated.

encounter longer cycles that seem like extended forms of bouncing: at least two electors i and i' produce a sequence of $q^{i,t}$ and $q^{i',t}$ values that form an extended cycle, again over a numerically small range of values; in every case we've observed the resulting closed figure is a rectangle; we have encountered such rectangles in which a single cycle occupies hundreds of iterations (see e.g. Figure 6). We suspect these, too, are numerical artifacts due to our imposing a positive lower bound for $k^{i,t}$, and we consider such solutions to be equilibria. We have not encountered more extensively ranging cycles, but we imagine such may occur, even if perhaps not with the current voting rules.

*** Figure 6 about here ***

Orbits out of equilibrium are interesting for helping to understand the dynamics and, possibly, to help when delineating domains of attraction for equilibria. We imagine the domains of attraction expressed in terms of the space of vectors of the q^i values of all electors are too complicated to map, in general, and we do not sketch a plan for such mapping here (cf. Siegel 2018). We illustrate a few trajectories just to give a taste of the lurking complexities.

Figure 7 shows phase plots for the expected vote counts n_j of the two parties that exhibit positive vote counts in equilibrium in each scenario shown in Figure 5. A small circle in each plot shows the initial vote count pair value. All four phase plots show expected vote counts that increase or decrease greatly—sometimes both—and that often spiral around. Of course n_j is not a fundamental dynamic parameter of the model—the fundamental dynamic parameters are the q^i. So the appearance that some orbits cross themselves in the figure is not problematic. It is fair to say the trajectories are at least moderately complicated, although of course they are far simpler than the complexities we would encounter with chaotic dynamics (Guckenheimer and Holmes 1986).

*** Figure 7 about here ***

The orbits of electors' mixed strategies may exhibit large oscillations, even the orbits of electors that in equilibrium end up with $q^i = 0$ or $q^i = 1$. Figure 8 shows orbits for five electors for each scenario shown in Figure 5. Electors are not matched across scenarios, rather in each case we show the orbits of any electors that in equilibrium have $q^i \in (0, 1)$, with the remaining electors chosen so that each of the five electors most prefers a distinct candidate. Orbits of the two parties that exhibit positive vote counts in equilibrium in each scenario are also shown. Perhaps most notable in all plots are the moments when some electors rapidly oscillate between $q^i = 0$ and $q^i = 1$. That such rapid and large oscillations occur while the aggregate counts vary only slightly reflects the fact that only a few electors are exhibiting oscillations. Sometimes electors' orbits move in what appears to be synchrony (albeit sometimes opposingly) with other of the displayed electors. We expect that many electors will exhibit such seemingly synchronized orbits, but we have yet to check this.

*** Figure 8 about here ***

4.1.4 Election Forensics with Simulated Data

To evaluate whether the kind of strategic behavior modeled here can trigger some of the statistical methods that are routinely used in election forensics, we generate 300 simulated pairs of vote counts using 300 runs of (14), each with expected electorate size $\mu = 1000$, utilities generated using $\underline{u}^i_j \sim \text{Beta}(\alpha_1, \alpha_2)$ as described in Section 4.1.1, costs compressed into $c^i \in (.015, .04)$ as described in Section 4.1.2 and initial values $q^{i,t_0} \sim$ Unif(0,1). For each run we use fresh draws of pseudorandom numbers. We take the two positive equilibrium expected vote counts from each run and consider their rounded values to be the "precinct" vote counts for two candidates in an election that we assess using election forensics statistical methods.[23] We should note that counts produced this way probably

[23] The first positive count is used for the first candidate, and the second positive count is used for the second candidate.

differ from those we would obtain if we simulated an electorate with an expected size of 300,000 that we divided into 300 precincts. We would need to design the precinct groupings, costs would need to be set differently, and other changes might be desirable. But the current simulated data support an initial quick-and-dirty check of whether, as Mebane (2013*a*, 2016) suggests, the kinds of statistics used in election forensics might be triggered by strategic behavior.

We observe that 300 is not a large sample size for election forensics analysis. With such a small sample size, digit-based tests lack power. But such a sample size, or smaller, is often presented for analysis as the precinct data from city or local elections. In any case, let's see.

Using the digit and unimodality tests implemented in the Election Forensics Toolkit[24] (Hicken and Mebane 2015) we obtain imprecise estimates for all of the digit tests and find no significant departures from unimodality (Table 3). The spikes test (Rozenas 2017)—a test on the proportions of the vote obtained by the winning candidate (here candidate 1)—also estimates that no fraud is present. But as Table 3 shows, the 2BL (second-digit mean) statistic for the second candidate (highlighted in red) has a confidence interval that does not include 4.187, which is the value expected according to the relevant Benford's Law-like distribution (see Mebane 2013*a*). As Mebane (2011, 2013*a*) suggests holds extensively in a variety of electoral systems and for a variety of strategies, so here also we find that strategic behavior triggers the second-digit test.

*** Table 3 about here ***

Estimates using the likelihood finite mixture model that was used in Mebane (2016) also suggest there is significant "incremental fraud" (Klimek et al. 2012), produced mainly by manufacturing votes from nonvoters. Table 4 reports the estimates. Statistical significance is assessed using a likelihood ratio test versus a specification that omits frauds: the likelihood ratio test statistic is 38.2, which we compare to a chi square distribution with four degrees of freedom ($p = 1.02\text{e-}07$). The interpetation that votes are being

[24]See `http://electionforensics.ddns.net:3838/test/`.

manufactured is based on the finite mixture model's α parameter, which is estimated to be $\hat{\alpha} = 11.0$. These "frauds," while statistically significant, are estimated to be rare. The probability that a precinct experiences incremental fraud is estimated to be $\hat{f}_{\mathrm{i}} = .0076$. That's about two precincts out of 300. In this instance, in line with the theoretical argument and speculations of Mebane (2016), strategic behavior masquerades as a bit of apparent fraud. Strategic behavior can create a kind of baseline buzz that signals of actual frauds need to exceed.

*** Table 4 about here ***

4.2 Imagined Future Extensions

The model of strategic abstention with wasted vote logic reflects important aspects of the strategic behavior that occur in elections, but many more kinds of strategies exist. Anytime each elector takes into account its anticipations of what some other electors will do, based on information that is at least approximately accurate and that is evaluated in a somewhat rational manner, what we call strategic behavior occurs. The type of strategic behavior implemented in our current ABMs is in a sense pristine, but of course the ABMs can be made noisier and sloppier. Also other types of strategic behavior are well known such as imitation, rational ignorance, threshold insurance, bandwagon dynamics and coalitions. We sketch extensions of the model of strategic abstention with wasted vote logic that relate to the latter two types of strategic behavior.

strategic "duty" Net costs are $c^i = C^i - D^i$. The bandwagon idea to make "duty" strategic is to define $D^i = \mathfrak{d}^i(z^i)$ for some function $\mathfrak{d}^i$ and argument z^i that is based on some summary of other's behavior or strategies. For instance, z^i might be a weighted sum of all others' mixed strategies $q^{i'}$ with weights that increase as utilities are more similar to the focal elector (e.g., for electors i and i' and utility vectors u^i and $u^{i'}$, the weight might be $w(i,i') = \langle u^i, u^{i'} \rangle / \sqrt{\|u^i\| \|u^{i'}\|}$): an elector is more prone to vote if electors like it are more likely to vote. The weights might also reflect a network imposed independent of utilities. Or weights—or z^i directly—may reflect how close an election outcome is expected to be. Such strategic duty may help overcome a limitation of the strategic abstention framework, which is that pivotal probabilities become so small in large electorates that turnout to cast an instrumental vote never occurs if net costs are positive. Strategic duty also suggests a variety of contagious voter suppression schemes it would be useful to simulate: e.g., force down q^i for a few electors and similar electors become less likely to vote.

coalitions Parties may form coalitions, which may mean that instead of each standing separately they offer a single coalition party alternative. If parties j and h form a coalition labeled $\mathfrak{C}(j,h)$, we can say electors form utilities for $\mathfrak{C}(j,h)$ by taking a weighted average of the utilities for the coalescing parties:

$$u^i_{\mathfrak{C}(j,h)} = \tau^i_{jh} u^i_j + (1 - \tau^i_{jh}) u^i_h \,, \quad \tau^i \in (0,1)\,.$$

Analogously for coalitions of more than two parties. Then if out of J_o "original" parties $J_{\mathfrak{D}}$ of the "original" parties are in $J_{\mathfrak{C}}$ coalitions, then $J = J_o - J_{\mathfrak{D}} + J_{\mathfrak{C}}$ parties contest the election based on electors having the derived utilities.

As described so far the coalition construction is not strategic. To make coalitions strategic make τ^i_{jh} a function of other electors' mixed strategies. Using $u^i_j \in [0,1]$ for all electors and all parties, define

$$\xi^i_j = \frac{\sum_{i' \neq i} (1 - u^{i'}_j) q^{i'}}{\tilde{\mu}_j}$$

and let $\tau^i_{jh} = .5[1 + (\xi^i_h - \xi^i_j)]$. If the other electors who are more likely to vote rate both j and h (for whom they cannot vote) equally then $u^i_{\mathfrak{C}(j,h)}$ is midway between u^i_j and u^i_h, otherwise $u^i_{\mathfrak{C}(j,h)}$ is closer to u^i_j if likely voters rate j as better than h, and

$u^i_{\mathfrak{C}(j,h)}$ is closer to u^i_h if h is better among likely voters than is j. Each elector thinks the coalition $\mathfrak{C}(j, h)$ is more similar to party j in $\mathfrak{C}(j, h)$ the more other electors who are likely voters support party j. It's as if a coalition's enthusiasts reveal where it really stands.

Some thought will be needed to extend this idea to coalitions that contain more than two parties.

5 Discussion

Using ABMs and a Poisson game information structure, we replicate basic equilibrum results from Demichelis and Dhillon (2010) and Cox (1994). Symmetric mixed strategy equilibria that Demichelis and Dhillon (2010) show are dynamically stable given their learning model based on a differential equation we find are dynamically stable using, effectively, difference equations. We find that Duvergerian equilibria such as Cox (1994) studies are dynamically stable in our system. We generalize Cox (1994) and Demichelis and Dhillon (2010) by simulating a model that combines strategic abstention with wasted vote logic: the model features a mixed strategy for abstention and a conditional pure strategy for the vote choice. We exhibit multiple equilibria in the combined model produced by different initial values for electors' abstention mixed strategies (compare Bouton and Gratton 2015).

Finding these equilibria requires careful attention to numerical details.[25] Using the Poisson game approach with Skellam distributions, pivotal probabilities are extremely small with large electorates. The expected electorate sizes we use here, $\mu = 400$ and $\mu = 1000$, are not large compared to the country-scale electorates that will ultimately concern us. So far we have been able to work effectively with double-precision arithmetic,

[25]Currently our simulation code is written in Python 3 (Python Core Team 2015) using GNU Parallel (Tange 2011), with analysis tools written in R (R Development Core Team 2011). Keeping track of the source code, conditions, arguments, orbits and equilibria across many variants and simulation runs depends on carefully executed database infrastructure. We use PostgreSQL (Stonebraker and Rowe 1986).

but when electorates are much larger use of extended precision numerical libraries will be important when pursuing models that rely on pivotal probabilities. Probably in very large electorates it will be both practically and theoretically reasonable to move away from a sharp focus on accurate and precise pivotal probabilities.

In our simulations we still encounter some anomalous (if numerically small) behavior that traces to numerical imprecision in implementing dynamics. Possibly we will need to abandon our approach using difference equations—that is, each agent updates at discrete time steps—even with adaptive learning rates. Adaptive learning rates make it tricky to argue that all agents are moving synchronously in time: each agent's time scale is interpretable only relative to its learning rates. But as long as, for election forensics purposes, we care only about equilibria and do not interpret orbits out of equilibrium, we think the loss of temporal comparability across agents is unimportant. But either because worse anomalies arise from the difference equation approach or because our focus changes—perhaps we will imagine elections occur before an equilibrium point is reached—we may move to a differential or difference-differential scenario. A difference-differential scenario may arise if we think that agents evolve in continuous time but "polls" that update their information about expected election outcomes occur at discrete moments.

We do not yet incorporate elector beliefs explicitly into the models. Agents also have fixed preferences. In the current implementations information is updated without reference to any prior beliefs—other than whatever beliefs are implicit in the specification of the current state—based on public information about the currently expected outcome of the election. We are considering whether to augment the models to make agents explicitly Bayesian, and we will investigate what happens when information is more limited. For example, agents may learn about the intentions only of their neighbors, in various senses of "neighbor." Or agents may be only adaptively rational.

It is remarkable that literally our first attempt generating a simulated electorate produces data that trigger some election forensics statistical methods. That 2BL is triggered generally matches findings reported by Mebane (2013*a*), but the simulation model's strategic mechanism lacks features such as ballot box disaggregations or district imbalances that Mebane (2006*a*,*b*, 2007, 2008, 2010*a*, 2012, 2013*a*) shows in simpler simulations can stimulate departures from the Benford's Law-like distribution. Perhaps the model of strategic abstention with wasted vote logic matches some aspects of the kinds of strategies involved in coordinating voting in the United States, threshold insurance in Germany and responses to coalitions in Mexico, all of which are associated with systematic patterns in the second-digit means (Mebane 2013*a*). But to clarify matters we'll want to develop more focused ABMs that more sharply represent such processes.

That the likelihood finite mixture model is triggered matches the findings of Mebane (2016), but we need to do more to explore the modality of the turnout and voting distributions in the simulated data and to identify what about the strategic mechanism enhances such multimodality.

In general our idea is to construct models for various election systems and strategies and calibrate how a variety of strategies affect various election forensics methods. If we can demarcate the patterns induced by strategic buzz, then signals genuinely triggered by frauds can stand out. In addition we can simulate various kinds of frauds. A grand vision is to use election forensics tools to extract features from election data, with the interpretation of those features guided by machine learning tools that have been trained using data simulated by ABMs. With ABMs we know exactly the conditions used to simulate data and exactly what every simulated elector does, so labelling training sets involves no ambiguity whatsoever. Such methods may help reduce tremendously the extent to which election forensics depends on a few stark cases and on a lot of more or less well informed interpretation.

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Table 1: Numbers of Electors Whose Equilibrium Mixed Strategies are not Zero or One

Number of Electors with $q^i \in (0,1)$	net costs: all positive, Beta parameters .5	1	10	net costs: some negative, Beta parameters .5	1	10
0	107	99	110	191	198	299
1	159	166	97	117	83	0
2	30	42	70	30	19	30
3	26	19	39	9	3	0
4	2	2	13	3	4	8
5	4	5	2	4	0	0
6	1	1	0	0	1	0
7	2	0	0	1	0	0
8	0	0	1	0	0	0
10	0	1	1	0	0	0
11	1	0	0	0	0	0

Note: frequency of runs for each costs-utility (-Beta parameter) condition shown in Figure 2 that have the indicated number of equilibrium mixed strategies q^i that are not either zero or one.

Table 2: Numbers of Electors Whose Equilibrium Mixed Strategies are in [.25,.75]

Number of Electors with $q^i \in [.25, .75]$	net costs: all positive, Beta parameters .5	1	10	net costs: some negative, Beta parameters .5	1	10
0	229	226	209	301	263	337
1	99	103	107	54	45	0
2	4	6	17	0	0	0

Note: frequency of runs for each costs-utility (-Beta parameter) condition shown in Figure 2 that have the indicated number of equilibrium mixed strategies $q^i \in [.25, .75]$.

Table 3: Digit and Unimodality Tests Using Simulated Data

Name	2BL	LastC	P05s	C05s	DipT	Obs
Turnout	4.46	4.593	.187	.21	.872	300
	(4.123, 4.78)	(4.24, 4.94)	(.143, .233)	(.16, .253)	–	
cand 1	4.25	4.41	.163	.2	.467	300
	(3.92, 4.593)	(4.104, 4.693)	(.117, .203)	(.157, .247)	–	
cand 2	4.53	4.35	.163	.207	.467	300
	(4.207, 4.85)	(3.993, 4.68)	(.123, .203)	(.16, .253)	–	

Note: "2BL," second-digit mean; "LastC," last-digit mean; "C05s," mean of variable indicating whether the last digit of the vote count is zero or five; "P05s," mean of variable indicating whether the last digit of the rounded percentage of votes for the referent party or candidate is zero or five; "DipT," p-value from test of unimodality; "Obs," number of polling station observations. Values in parentheses are nonparametric bootstrap confidence intervals.

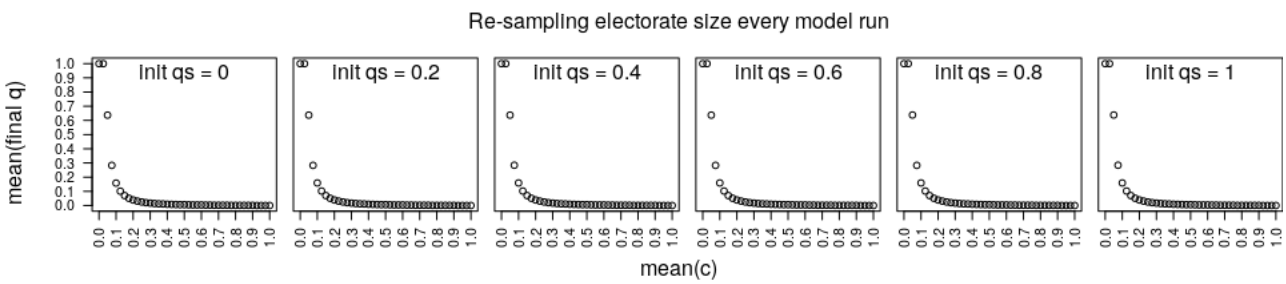
Table 4: Finite Mixture Model Parameter Estimates Using Simulated Data

Election	$\hat{f}_{\mathrm{i}}$	$\hat{f}_{\mathrm{e}}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\tau}$	$\hat{\nu}$	LR	n
data simulated using (14)	.00759	0	11.0	.657	.372	.497	38.2	300

Note: LR is the likelihood ratio test statistic for the hypothesis that there are no frauds (i.e., that $f_{\mathrm{i}} = f_{\mathrm{e}} = 0$). n is the number of simulated precinct observations.

Figure 1: Probability of voting (q) versus cost of voting (c)

(a) identical expected electorate sizes ($\mu = 400$)



(b) identical electorate sizes ($N_e = 368$)

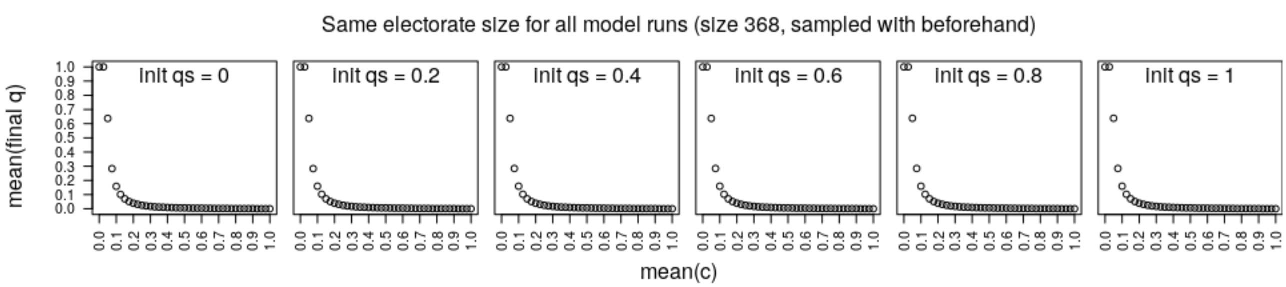
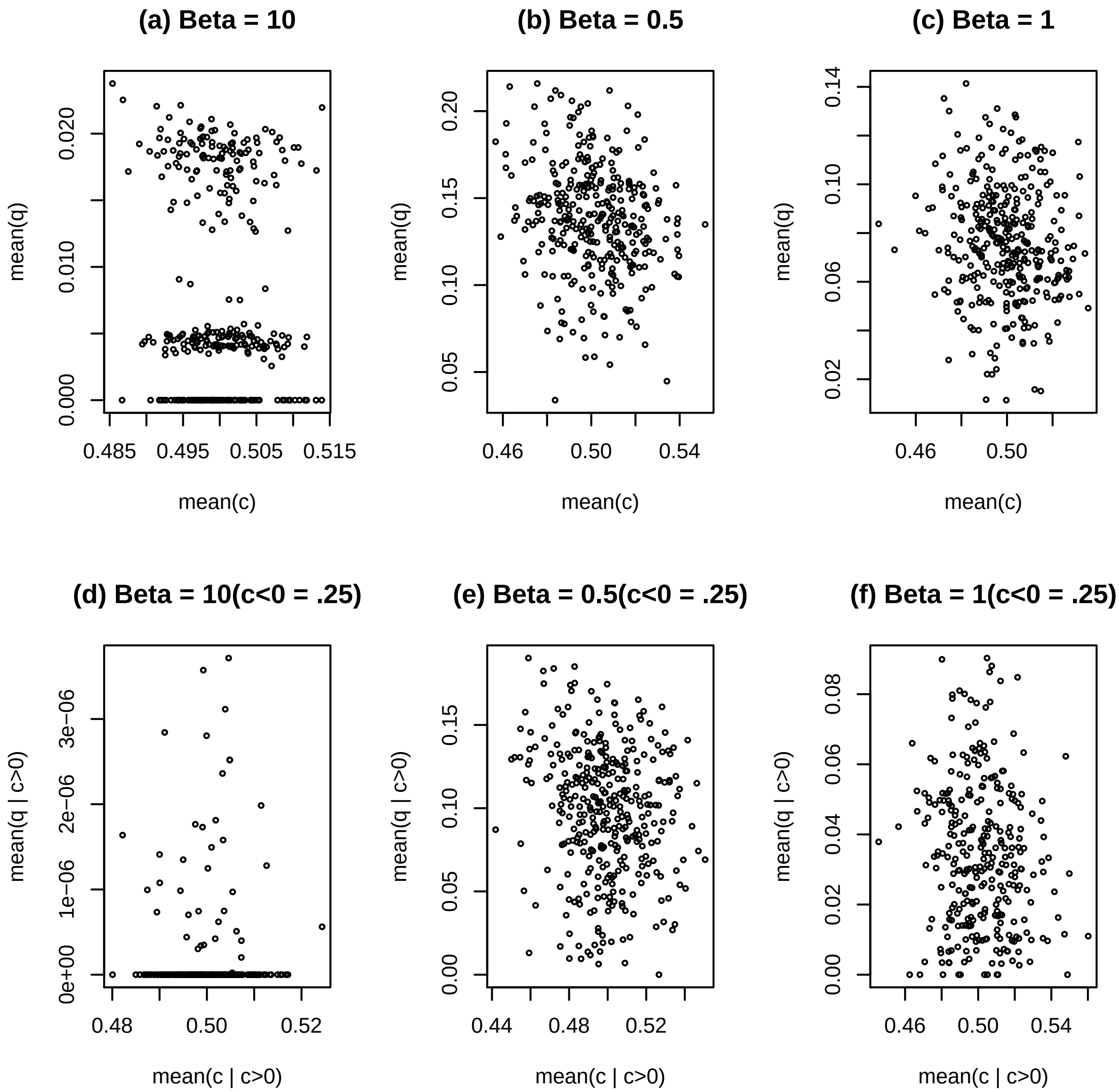
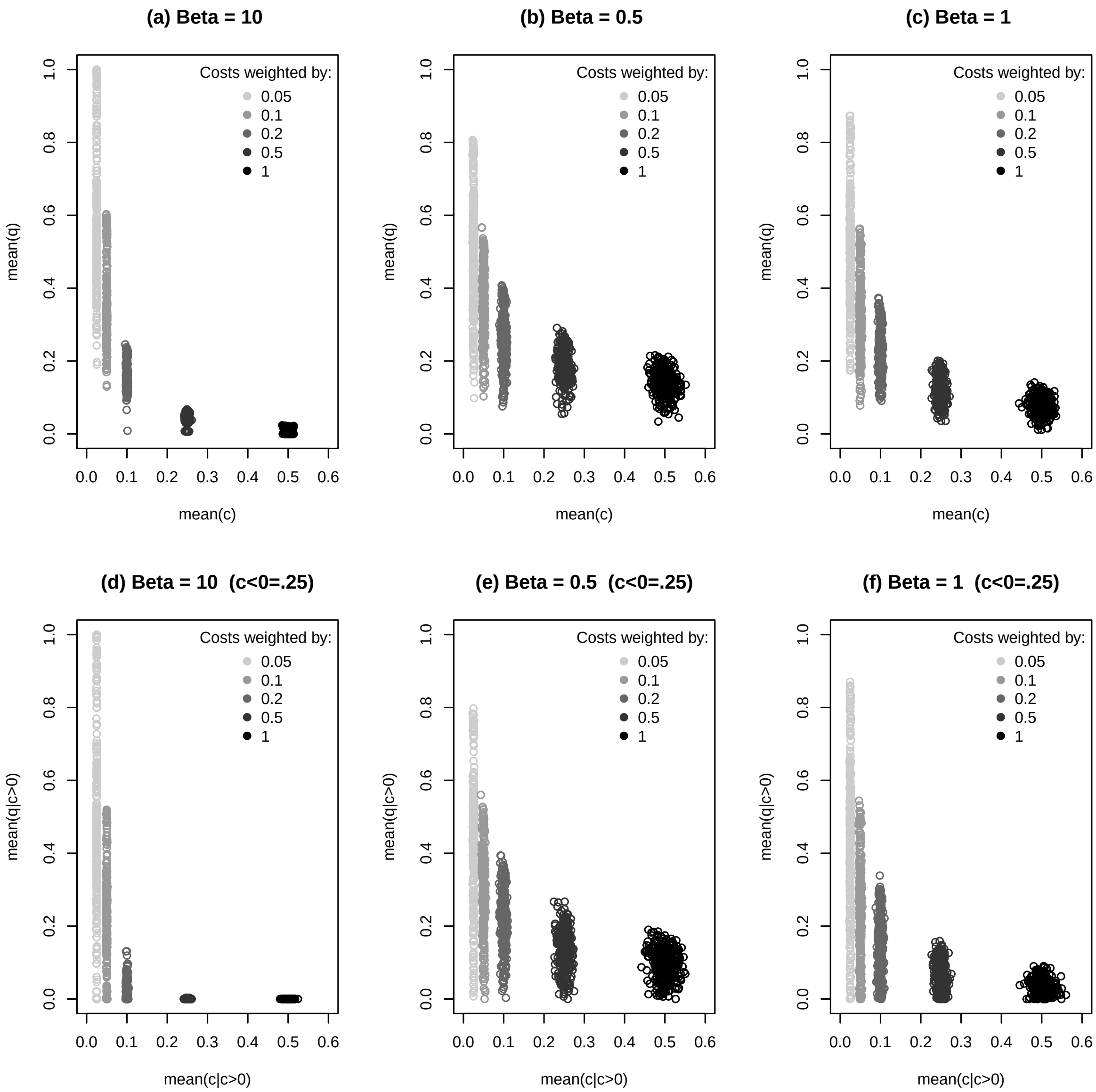


Figure 2: Strategic Abstention-Wasted Vote Model: Mean Costs and Strategies



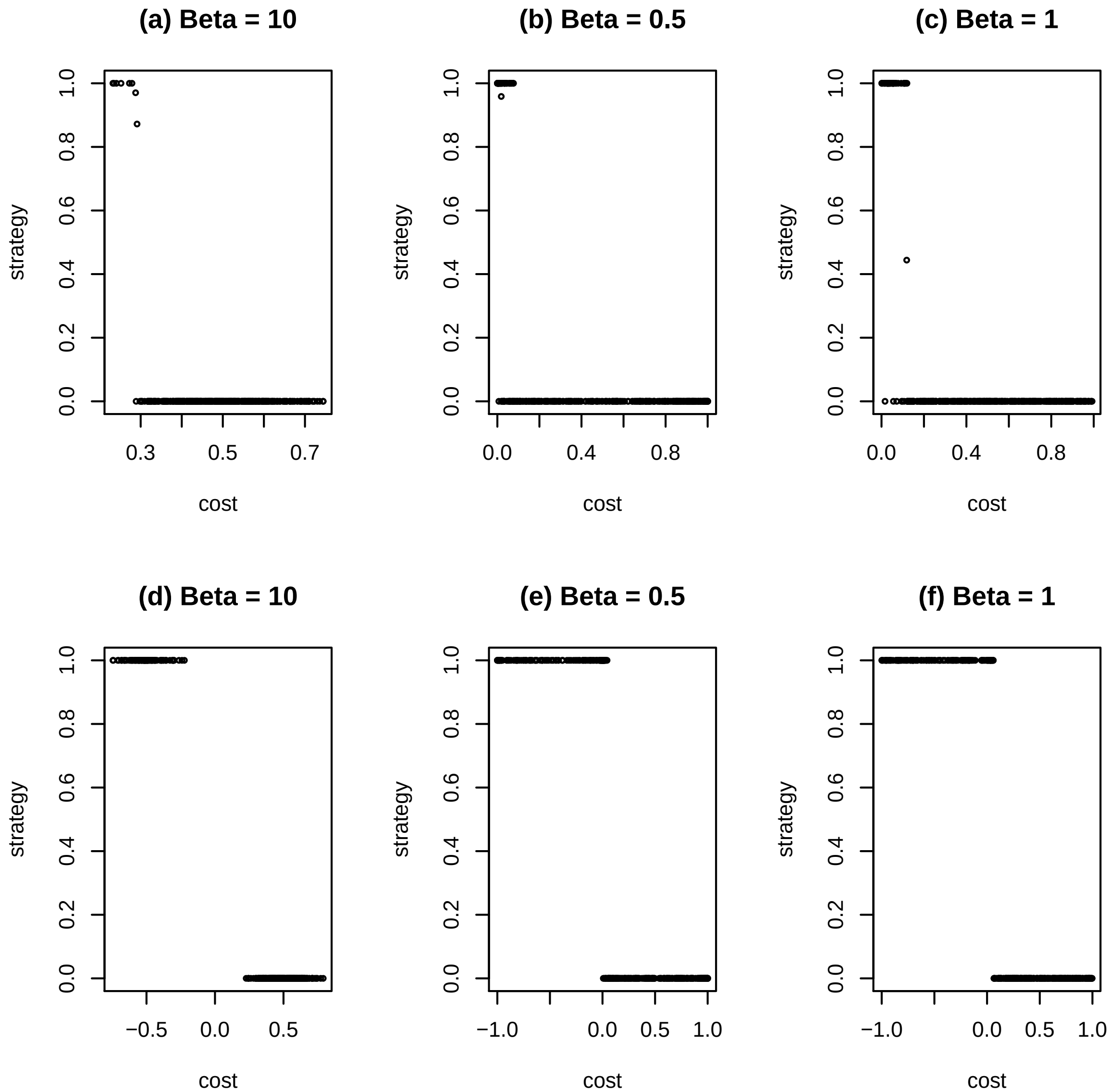
Note: mean net costs and mean mixed strategies for electors that have positive net costs in simulations of (14) using $\mu = 400$ for various net cost distributions. All net costs positive: (a) costs from Beta(10,10), $n = 333$; (b) costs from Beta(.5,.5), $n = 332$; (c) costs from Beta(1,1), $n = 335$. About 1/4 of net costs made negative: (d) costs from Beta(10,10), $n = 337$; (e) costs from Beta(.5,.5), $n = 355$; (f) costs from Beta(1,1), $n = 308$.

Figure 3: Strategic Abstention-Wasted Vote Model: Mean Costs and Strategies, Rescaled Costs



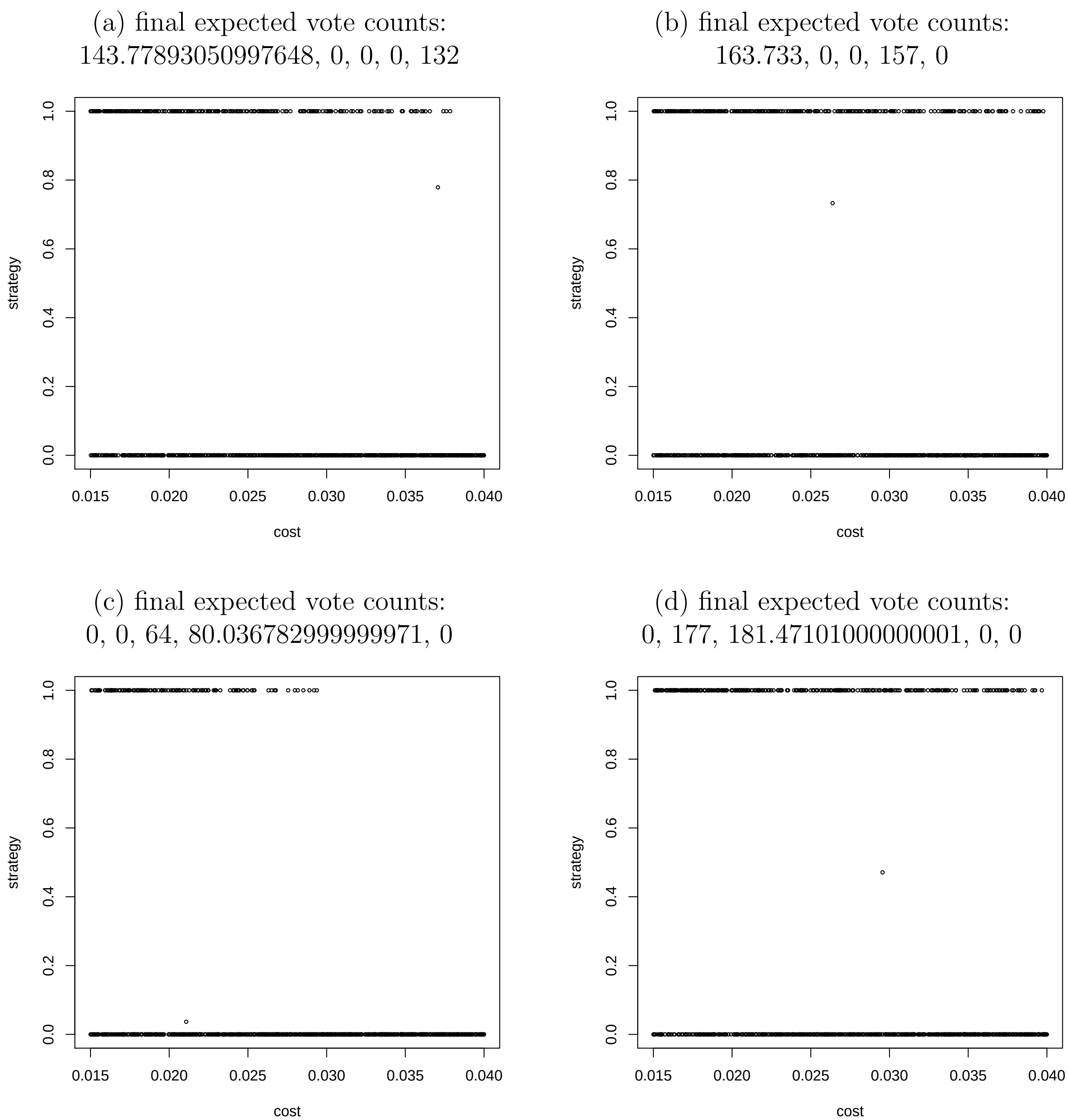
Note: mean net costs and mean mixed strategies for electors that have positive net costs in simulations of (14) using $\mu = 400$ for various net cost distributions. All net costs positive, rescaled respectively by $c^i = s\underline{c}^i$ for $s \in \{.05, .1, .2, .5, 1\}$ where $\underline{c}^i$ is randomly generated as follows: (a) costs from Beta(10,10), $n \in (307, 328, 337, 340, 333)$; (b) costs from Beta(.5,.5), $n \in (349, 311, 349, 330, 332)$; (c) costs from Beta(1,1), $n = \in (344, 361, 314, 330, 335)$. About 1/4 of rescaled net costs made negative: (d) costs from Beta(10,10), $n = \in (350, 335, 308, 342, 337)$; (e) costs from Beta(.5,.5), $n = \in (320, 325, 346, 318, 355)$; (f) costs from Beta(1,1), $n = \in (330, 340, 346, 340, 308)$.

Figure 4: Strategic Abstention-Wasted Vote Model: Elector Costs and Strategies



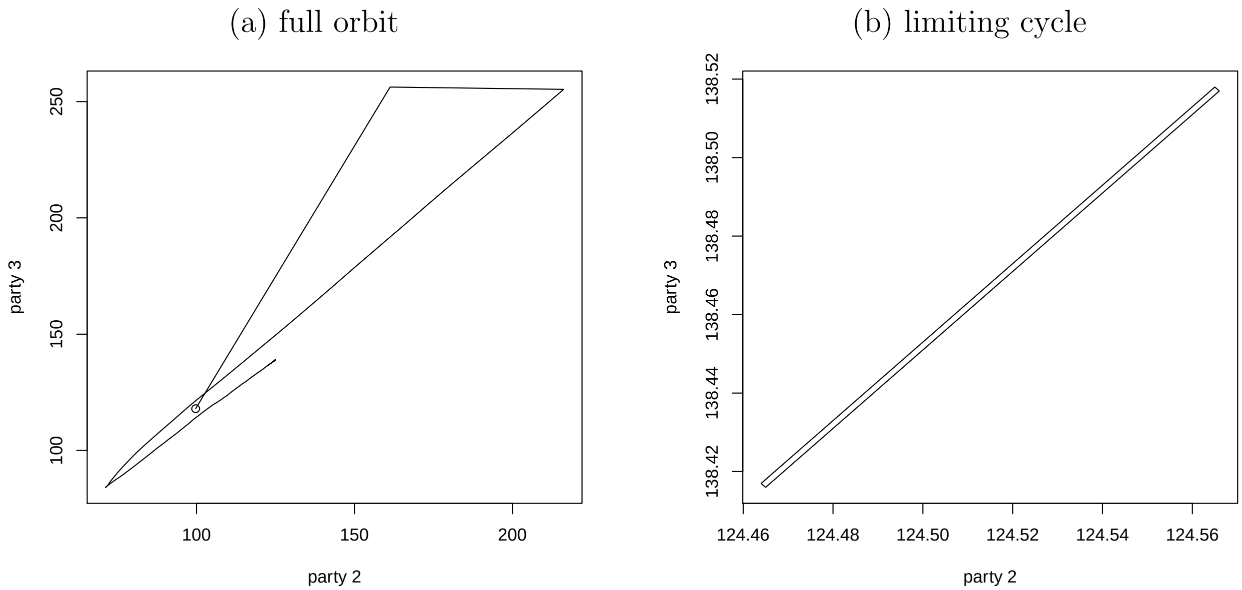
Note: net costs and mixed strategies for electors in single simulations of (14) using $\mu = 400$ for each net cost distribution. All net costs positive: (a) costs from Beta(10,10); (b) costs from Beta(.5,.5); (c) costs from Beta(1,1). About 1/4 of net costs made negative: (d) costs from Beta(10,10); (e) costs from Beta(.5,.5); (f) costs from Beta(1,1).

Figure 5: Strategic Abstention-Wasted Vote Model: Elector Costs and Strategies, Multiple Equilibria



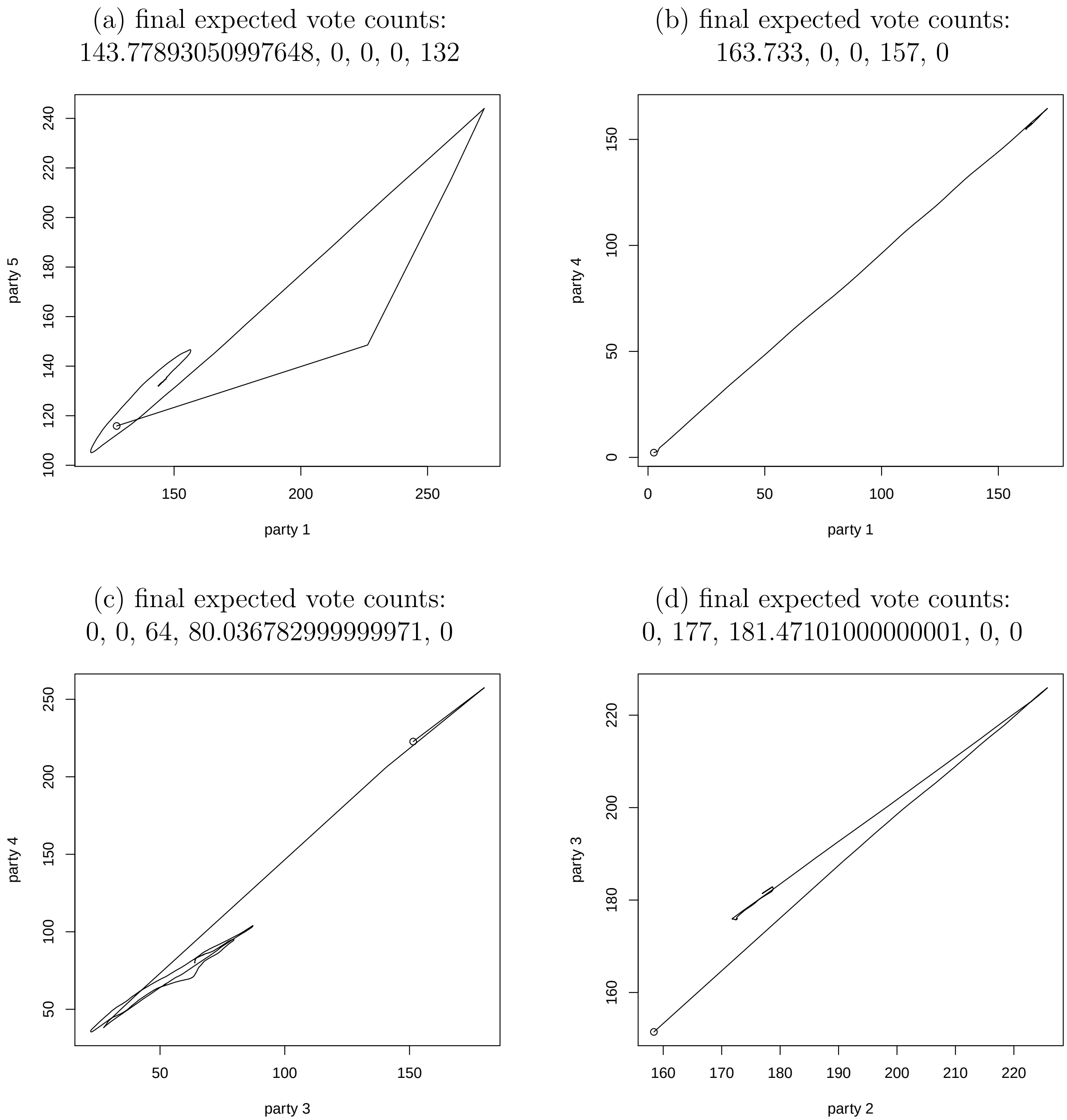
Note: net costs and mixed strategies for electors in four simulations of (14) ($\mu = 1000$, $N_e = 1017$, $J = 5$) with identical utilities across simulations for a compressed net cost distribution. All net costs positive: $c^i = .015 + .025\underline{c}^i$ where $\underline{c}^i \sim \text{Beta}(1, 1)$.
Sincere vote (first-preference) counts: 251, 160, 153, 225, 228.

Figure 6: Strategic Abstention-Wasted Vote Model: Expected Vote Count Phase Plots



Note: phase plots of expected vote counts for the two parties that have positive vote counts in Duvergerian equilibrium in a simulation of (14) ($\mu = 1000$, $J = 5$). In (a) the initial expected vote count values are indicated using a circular point. (b) shows the limiting cycle (cycle length 205 iterations); the orbit travels clockwise around the box.

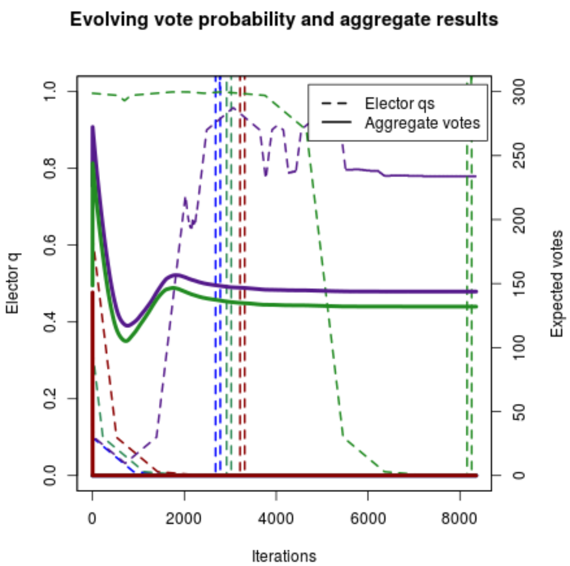
Figure 7: Strategic Abstention-Wasted Vote Model: Expected Vote Count Phase Plots, Multiple Equilibria



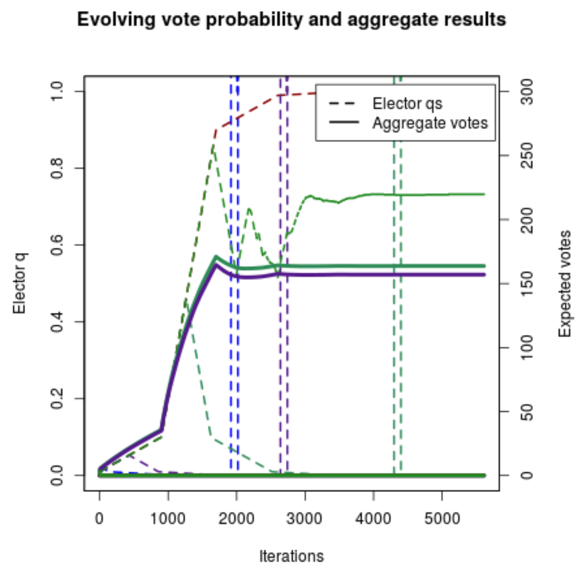
Note: phase plots of expected vote counts for the two parties that have positive vote counts in Duvergerian equilibrium in four simulations of (14) ($\mu = 1000$, $N_e = 1017$, $J = 5$) with identical utilities across simulations for a compressed net cost distribution. The initial expected vote count values are indicated using circular points. All net costs positive:
$c^i = .015 + .025\underline{c}^i$ where $\underline{c}^i \sim \text{Beta}(1, 1)$.
Sincere vote (first-preference) counts: 251, 160, 153, 225, 228.

Figure 8: Strategic Abstention-Wasted Vote Model: Mixed Strategy Orbits, Multiple Equilibria

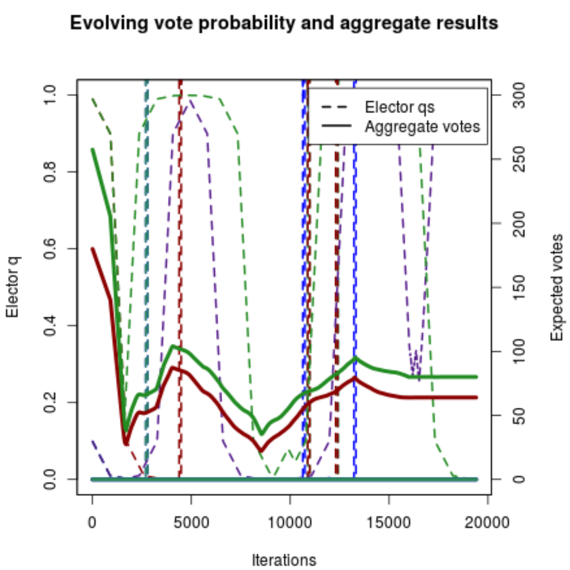
(a) final expected vote counts:
143.77893050997648, 0, 0, 0, 132



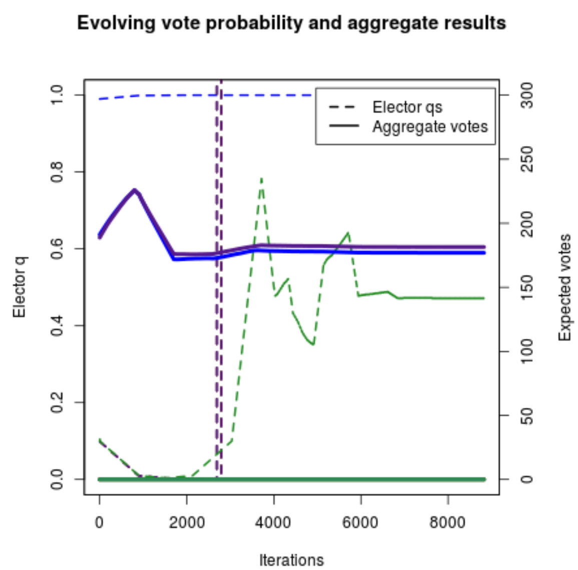
(b) final expected vote counts:
163.733, 0, 0, 157, 0



(c) final expected vote counts:
0, 0, 64, 80.036782999999971, 0



(d) final expected vote counts:
0, 177, 181.47101000000001, 0, 0



Note: orbits of mixed strategies for five electors in four simulations of (14) ($\mu = 1000$, $N_e = 1017$, $J = 5$) with identical utilities across simulations for a compressed net cost distribution. All net costs positive: $c^i = .015 + .025\underline{c}^i$ where $\underline{c}^i \sim \text{Beta}(1, 1)$. The five electors in each plot have distinct most-preferred candidates.