Notes on calculus in more than one real variable

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2020

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Introduction

These notes on the differential and integral calculus of functions $f : \mathbb{R}^n \to \mathbb{R}^m$ are intended as a ***reminder document only***. The goal is to state ideas and rules that I want to be able to quickly look up and remember, and to present examples so that I quickly remember how to apply them. Since they are only meant to be used as a fact sheet, I omit proofs and explanatory illustrations. For anything more than that, please see the cited texts themselves.

These notes in particular have a bit of backstory. I wrote them in January — April 2020 while taking the last class in the real analysis sequence at the University of Michigan. That infamous semester, of course, completely disintegrated about halfway through, right around the midterm when I usually polish up the first half of my course notes. So section 2, on multivariable differential calculus, is unfortunately extremely slight. In contrast, by the time we reached section 4, everything was locked down, and I (was lucky enough that I) had nothing better to do than sit around and take thorough notes on integral calculus. So the second half of the class takes up much more than half of the notes.

While I was doing this, something really weird happened: I quickly realised that I love the theory of multivariate integrals. That was very bizarre to me because I thought I already knew a lot about it, and I really hated it: as a physics major all I did was take messy integrals of disgusting functions 24/7. But that's probably why I thought it was so terrible. Go figure.

Also, though the first half of the notes are much shorter, I have luckily already written teaching notes on most of the topics that we covered in the first half of the class: see my teaching notes for POLSCI 598 at the University of Michigan, which was mostly a class on multivariable differential calculus.

Finally, as always, I know that there are typos in here, some of them bad. I'll be grateful if you notify me when you spot one. Let me know at sbaltz {at} umich {dot} edu.

1 Euclidean space and linear mappings

1.1 The vector space $\mathbb{R}^n$

Definition 1. [2: p. 2] A **metric space** (X, d) is a nonempty set X together with a function $d : X \times X \to \mathbb{R}$ which satisfies the following axioms $\forall\ x, y, z \in X$:

$\hookrightarrow$ $d(x, y) = d(y, x)$

$\hookrightarrow$ $d(x, z) \leq d(x, y) + d(y, z)$

$\hookrightarrow$ $d(x, x) = 0$

$\hookrightarrow$ $x \neq y \implies d(x, y) \neq 0$

The function d is called a **metric** on X.

Definition 2. [2: p. 2] Let (X, d) be a metric space and $\{x_j\}$ a sequence in X, $x \in X$. We say that $\{x_j\}$ **converges** to x if, for any $\epsilon > 0$, there exists an $N \in \mathbb{R}$ such that $d(x_j, x) < \epsilon,\ \forall j > N$

Definition 3. [2: p. 2] Let S be a subset of a metric space (X, d). A point $x \in X$ is called a **limit point** of S if there is a sequence $\{x_n\}_{n=1}^{\infty}$ in S such that $x_n \neq x$ for any $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$

Definition 4. [2: p. 2] Let S be a subset of a metric space (X, d). The **closure** of S, denoted $\bar{S}$, is the set of all limit points of S. S is called **closed** if S contains all its limit points.

Definition 5. [2: p. 2] Let (X, d) be a metric space. The **open ball** (or **open neighbourhood**) with center $x \in X$ and radius $r > 0$ is defined as

$$B_r(x) = \{y \in X : d(y, x) < r\}$$

Similarly, the **closed ball** (or **closed neighbourhood**) with center $x \in X$ and radius $r > 0$ is defined as

$$B_r(x) = \{i \in X : d(y, x) \leq r\}$$

Definition 6. [2: p. 3] Let S be a subset of a metric space (X, d). A point $x \in S$ is called an **interior point** of S if there exists some $\epsilon > 0$ such that $B_\epsilon(x) \subset S$. The set of interior points is called **the interior** of S.

Definition 7. [2: p. 3] Let S be a subset of a metric space (X, d). S is called **open** if every point in S is an interior point of S.

Theorem 1. *[2: p. 3] Let S be a subset of a metric space (X, d). Then S is open iff $X \setminus S$ is closed.*

Definition 8. [2: p. 6] A set $S \subseteq \mathbb{R}^n$ is called **compact** if every sequence $\{x_n\}$ in S has a subsequence which has a limit in S.

1.2 Subspaces of $\mathbb{R}^n$

Definition 9. [2: p. 4] A **vector space** over $\mathbb{R}$ is a set with a function (called addition) mapping $V \times V \to V$, denoted

$$(\mathbf{x}, \mathbf{y}) \to \mathbf{x} + \mathbf{y}$$

as well as a function (called scalar multiplication) mapping $\mathbb{R} \times V \to V$, denoted

$$(a, \mathbf{x}) \to a\mathbf{x}$$

such that the following properties hold for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $a, b \in \mathbb{R}$:

- $\hookrightarrow$ $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$
- $\hookrightarrow$ $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
- $\hookrightarrow$ $\exists \mathbf{1} \in V$ so $1\mathbf{x} = x$
- $\hookrightarrow$ $\exists \mathbf{0} \in V$ so $\mathbf{0} + \mathbf{x} = \mathbf{x}$
- $\hookrightarrow$ $\mathbf{x} + (-1\mathbf{x}) = \mathbf{0}$
- $\hookrightarrow$ $(ab)\mathbf{x} = a(b\mathbf{x})$
- $\hookrightarrow$ $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$
- $\hookrightarrow$ $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + \mathbf{y}$

Definition 10. [2: p. 4] A **subspace** of vector space V is a subset W of V such that W is itself a vector space with the same operations.

Definition 11. [2: p. 4] Let V be a vector space over $\mathbb{R}$ A function $\phi : V \to \mathbb{R}$ is called a **norm** on V if the following hold $\forall$ $\mathbf{x}, \mathbf{y} \in X$ and $a \in \mathbb{R}$:

- $\hookrightarrow$ $\phi(\mathbf{x}+\mathbf{y}) \leq \phi(\mathbf{x}) + \phi(\mathbf{y})$
- $\hookrightarrow$ $\phi(a\mathbf{x}) = |a|\phi(\mathbf{x})$
- $\hookrightarrow$ $\mathbf{x} \neq \mathbf{0} \implies \phi(\mathbf{x}) \neq 0$

1.3 Inner products and orthogonality

Definition 12. [2: p. 5] An **inner product** on the vector space V is a function $V \times V \to \mathbb{R}$ which associated each pair $(\mathbf{x}, \mathbf{y})$ of vectors in V a real number $\langle \mathbf{x}, \mathbf{y} \rangle$, and satisfies the following conditions:

- $\hookrightarrow$ $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$
- $\hookrightarrow$ $\langle a\mathbf{x}, \mathbf{y} \rangle = a\langle \mathbf{x}, \mathbf{y} \rangle$
- $\hookrightarrow$ $\langle \mathbf{x}+\mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
- $\hookrightarrow$ $\mathbf{x} \neq \mathbf{0} \implies \langle \mathbf{x}, \mathbf{x} \rangle > 0$

It follows that $\langle \mathbf{x}, \mathbf{0} \rangle = \langle \mathbf{0}, \mathbf{x} \rangle = 0$.

Definition 13. [2: p. 5] The vectors $\mathbf{x}$ and $\mathbf{y}$ in a vector space V are **orthogonal** iff $\langle \mathbf{x}, \mathbf{y} \rangle = 0$

1.4 Limits and continuity

Definition 14. [2: p. 6] A sequence $\{x_n\}$ in a metric space is called a **Cauchy sequence** if, for every $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}$ such that $n, m > N$, $d(x_n, x_m) < \epsilon$.

Theorem 2. *[2: p. 6] If a sequence converges, then it is a Cauchy sequence.*

Definition 15. [2: p. 6] A subset S of a metric space (X, d) is **complete** if every Cauchy sequence in S has a limit in S. A normed vector space that is complete under a metric induced by a norm is called a **Banach space**. An inner product space that is complete under a metric induced by its inner product is called a **Hilbert space**.

Definition 16. [2: p. 6] A function $f : X_1 \to X_2$ from a metric space (X_1, d_1) to a metric space (X_2, d_2) is **continuous** at a point $x_0 \in X_1$ if, for each $\epsilon > 0$, there exists some $\delta > 0$ such that, for all $x \in X_1$,

$$d_1(x, x_0) < \delta \implies d_2(f(x), f(x_0)) < \epsilon$$

The function f is called **continuous** if it is continuous at every point in X_1.

2 Multivariable differential calculus

Remark 1. [1: p. 56-57] Recall from single-variable differential calculus that if the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a, then the tangent line at $(a, f(a))$ to the graph $y = f(x)$ in $\mathbb{R}^2$ is the straight line with the equation

$$y - f(a) = f'(a)(x - a)$$

The crucial observation is that the right-hand side is a linear approximation to the actual difference $f(x) - f(a)$ between the values that f outputs at x and at a. Set $h \equiv x - a$, $\Delta f_a(h) = f(a + h) - (a)$, and $df_a(h) = f'(a)h$. The linear mapping $df_a : \mathbb{R} \to \mathbb{R}$, defined by $df_a(h) = f'(a)h$, is called the *differential* of f at a; this is the linear mapping $\mathbb{R} \to \mathbb{R}$ whose matrix is the *derivative* $f'(a)$ of f at a. Of course, in this case because the mapping takes one real number onto one real number, that matrix is simply the real number. This is useful because when h is small, the linear change $df_a(h)$ is a good approximation to the actual change $\Delta f_a(h)$: precisely,

$$\lim_{h \to 0} \frac{\Delta f_a(h) - df_a(h)}{h} = \lim_{h \to 0} \frac{f(a + h) - f(a) - f'(a)h}{h} = 0$$

With these single-variable ideas in mind, we develop an analogous idea for functions $f : \mathbb{R}^n \to \mathbb{R}^m$, with the notion that such a function is differentiable at point $\mathbf{a}$ iff it has near $\mathbf{a}$ an approriate linear approximation $df_{\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}^m$. Here $df_{\mathbf{a}}$ is the differential of f at $\mathbf{a}$, and its $m \times n$ matrix is called the derivative of f at $\mathbf{a}$. This will preserve both geometric intuition and

2.1 Curves in $\mathbb{R}^m$

Remark 2. [1: p. 58] Any function $f : \mathbb{R} \to \mathbb{R}^m$ is differentiable at a point a if and only if each of its coordinate functions $f_1, \ldots, f_m$ is differentiable at a, in which case

$$f' = (f_1', \ldots, f_m')$$

Theorem 3. *[1: p. 60] [2: p. 13] The mapping $f : \mathbb{R} \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}$ iff $\exists$ a linear mapping $L : \mathbb{R} \to \mathbb{R}^m$ satisfying*

$$\lim_{h \to 0} \frac{f(a + h) - f(a) - L(h)}{h} = \mathbf{0}$$

When that linear mapping exists, we call it $L(h) \equiv df_a(h) = hf'(a)$, the differential of f at a.

Theorem 4. *[1: p. 58] Let f and g be mappings from $\mathbb{R}$ to $\mathbb{R}^m$, and $\phi : \mathbb{R} \to \mathbb{R}$, all differentiable. Then*

$\hookrightarrow (f+g)' = f' + g'$
$\hookrightarrow (\phi f)' = \phi' f + \phi f'$
$\hookrightarrow (f \cdot g)' = f' \cdot g + g' \cdot f$
$\hookrightarrow (f \circ \phi)'(t) = \phi'(t) f'(\phi(t))$

Remark 3. [1: p. 60] The derivative vector $f'(a)$ is, as a column vector, the matrix of the linear mapping df_a:

$$df_a(h) = hf'(a) = \begin{pmatrix} f_1'(a) \\ \vdots \\ f_m'(a) \end{pmatrix} (h)$$

Definition 17. [2: p. 14] For a curve $f : \mathbb{R} \to \mathbb{R}^m$, the **tangent line** at $f(a)$ is the straight line which passes through $f(a)$ and is parallel to $f'(a)$, so the tangent line at point a is an affine mapping $T_a : \mathbb{R} \to \mathbb{R}^m$ given by

$$T_a(t) = f(a) + t(f'(a)) = (f_1(a), \ldots, f_m(a)) + (f_1'(a), \ldots, f_m'(a))(t-a)$$

With $t = a$, $T_a(a) = f(a)$ is exactly the point where the tangent line touches the curve.

2.2 Directional derivatives and the differential

Remark 4. [1: p. 63] The definition of a derivative of a function of a single variable was motivated by idea of defining a tangent line to a curve. In the case of multiple variables, we care about an analogous problem: we hope to define a tangent plane to a surface. The idea is that this plane should consist of the set of all straight lines which are tangent to the curves in the surface at a given point.

Definition 18. [1: p. 65] Given $\mathbf{v} \in \mathbb{R}^n$, a point $\mathbf{a} \in \mathbb{R}^n$, and a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, the **directional derivative with respect to v of** F **at a** is

$$D_{\mathbf{v}} F(\mathbf{a}) \equiv \lim_{h \to 0} \frac{F(\mathbf{a} + h\mathbf{v}) - F(\mathbf{a})}{h}$$

Remark 5. [1: p. 65] Of particular interest are the directional derivatives of $F : \mathbb{R}^n \to \mathbb{R}^m$ with respect to the standard unit basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$, which are called the "partial derivatives" of F at $\mathbf{a}$

Definition 19. [1: p. 66] The ith **partial derivative** of $F : \mathbb{R}^n \to \mathbb{R}^m$ at $\mathbf{a}$ is defined by

$$D_i F(\mathbf{a}) = \frac{\partial F}{\partial x_i}(\mathbf{a}) = D_{\mathbf{e}_i} F(\mathbf{a})$$

Remark 6. [1: p. 66] With $\mathbf{a} = (a_1, \ldots, a_n)$,

$$D_i F(\mathbf{a}) = \lim_{h \to 0} \frac{F(\mathbf{a} + h\mathbf{e}_i) - F(\mathbf{a})}{h}$$

$$D_i F(\mathbf{a}) = \lim_{h \to 0} \frac{F(a_1, \ldots, a_i + h, \ldots, a_n) - F(a_1, \ldots, a_i, \ldots, a_n)}{h}$$

Theorem 5. *[1: p. 68] If $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a}$, then the directional derivative $D_\mathbf{v} F(\mathbf{a})$ exists for all $\mathbf{V} \in \mathbb{R}^n$, and*

$$D_\mathbf{v} F(\mathbf{a}) = dF_\mathbf{a}(\mathbf{v})$$

Theorem 6. *[1: p. 69] If $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a}$, and $\mathbf{v} = (v_1, \ldots, v_n)$, then*

$$D_\mathbf{v} F(\mathbf{a}) = \sum_{j=1}^{n} v_j D_j F(\mathbf{a})$$

which relates the directional derivative of F to its partial derivatives.

Theorem 7. *[1: p. 70] The **gradient vector** of a differentiable real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is given by*

$$\nabla f(\mathbf{a}) = (D_1 f(\mathbf{a}), \ldots, D_n f(\mathbf{a}))$$

Remark 7. [1: p. 70] Note

$$D_\mathbf{v} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}$$

Lemma 8. *[1: p. 71] The mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a}$ iff each of its component functions $F^1, \ldots, F^m$ are, and*

$$dF_\mathbf{a} = (dF^1_\mathbf{a}, \ldots, dF^m_\mathbf{a})$$

Theorem 9. *[1: p. 72] If $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a}$, then the matrix $F'(\mathbf{a})$ of $dF_\mathbf{a}$ is*

$$F'(\mathbf{a}) = (D_j F^i(\mathbf{a}))$$

$$F'(\mathbf{a}) = \begin{pmatrix} \frac{\partial F^1}{\partial x_1} & \cdots & \frac{\partial F^1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x_1} & \cdots & \frac{\partial F^m}{\partial x_n} \end{pmatrix}$$

Definition 20. [1: p. 72] The mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is called **continuously differentiable** at $\mathbf{a}$ if the partial derivatives $D_1 F, \ldots, D_n F$ all exist at each point of some open set containing $\mathbf{a}$, and are continuous at $\mathbf{a}$.

Theorem 10. *[1: p. 72] If F is continuously differentiable at $\mathbf{a}$, then F is differentiable at $\mathbf{a}$.*

Theorem 11. *[1: p. 76] The chain rule: Let U and V be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. If the mappings $F : U \to \mathbb{R}^m$ and $G : V \to \mathbb{R}^k$ are differentiable at $\mathbf{a} \in U$ and $F(\mathbf{a}) \in V$ respectively, then their composition $H = G \circ F$ is differentiable at $\mathbf{A}$, and*

$$dH_{\mathbf{a}} = dG_{F(\mathbf{a})} \circ dF_{\mathbf{a}}$$

Therefore in terms of derivatives,

$$H'(\mathbf{a}) = G'(F(\mathbf{a})) \cdot F'(\mathbf{a})$$

2.3 Quadratic forms, Taylor's formula, and critical points

Theorem 12. *[1: p. 92] Let f and g be continuously differentiable functions on $\mathbb{R}^2$. Suppose that f attains its maximum or minimum value on the zero set S of g at the point $\mathbf{p}$ where $\nabla g(\mathbf{p}) \neq \mathbf{0}$. Then*

$$\nabla f(\mathbf{p}) = \lambda \nabla g(\mathbf{p})$$

for some number λ. That number λ is called a "Lagrange multiplier".

Definition 21. [2: p. 28] Write $f : \mathbb{R}^n \to \mathbb{R}$ up to the 2nd degree Taylor polynomial:

$$f(\mathbf{a} + \mathbf{h}) = f(\mathbf{a}) + \nabla f'(\mathbf{a}) \cdot \mathbf{h} + \frac{1}{2} D^2_{\mathbf{h}} f(\mathbf{a}) + R_2(\mathbf{h})$$

then assume that $\mathbf{a}$ is a critical point, so that $\nabla f(\mathbf{a}) = \mathbf{0}$, so that

$$f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) = \frac{1}{2} D^2_{\mathbf{h}} f(\mathbf{a}) + R_2(\mathbf{h}) = q(\mathbf{h}) + R_2(\mathbf{h})$$

We call $q(\mathbf{h})$ the **quadratic form** of f at the critical point $\mathbf{a}$.

Recall 1. A matrix is called each of the following if all of its eigenvalues λ satisfy:
Positive definite iff $\lambda > 0$
Positive semi-definite iff $\lambda \geq 0$
Negative definite iff $\lambda < 0$
Negative semi-definite iff $\lambda \leq 0$

Theorem 13. *[1: p. 131] Taylor's formula in several variables: if f is a real-valued function of class $\mathcal{C}^{k+1}$ on an open set containing the line segment L from $\mathbf{a}$ to $\mathbf{a}+\mathbf{h}$ then there exists a point $\xi \in L$ such that*

$$R_K(\mathbf{h}) = \frac{D_{\mathbf{h}}^{k+1}(\xi)}{(k+1)!}$$

so

$$f(\mathbf{a}+\mathbf{h}) = \sum_{r=0}^{k} \frac{D_h^r f(\mathbf{a})}{r!} + \frac{D_{\mathbf{h}}^{k+1} f(\xi)}{(k+1)!}$$

Theorem 14. *[1: p. 135] If $f : \mathbb{R}^n \to \mathbb{R}$ is of class $\mathcal{C}^{k+1}$ in a neighbourhood of $\mathbf{a}$, and Q is a polynomial of degree k such that*

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x}) - Q(\mathbf{x}-\mathbf{a})}{|\mathbf{x}-\mathbf{a}|^k} = 0$$

then Q is the kth degree Taylor polynomial of f at $\mathbf{a}$.

Theorem 15. *[1: p. 138] Let f be of class $\mathcal{C}^3$ in a neighbourhood of the critical point $\mathbf{a}$. Then f has*

- ↪ *a local minimum at $\mathbf{a}$ if its quadratic form $q(\mathbf{h})$ is positive-definite*
- ↪ *a local maximum at $\mathbf{a}$ if $q(\mathbf{h})$ is negative-definite*
- ↪ *neither if $q(\mathbf{h})$ is nondefinite*

Theorem 16. *[1: p. 142] If the quadratic form q attains its maximum or minimum value on S^{n-1} at the point $\mathbf{v} \in S^{n-1}$, then there exists a real number λ such that*

$$L(\mathbf{v}) = \lambda \mathbf{v}$$

*where L is the linear mapping associated with q. Then $\mathbf{v}$ is called an **eigenvector** of L and λ is its associated **eigenvalue**.*

Theorem 17. *[1: p. 148] If q is a quadratic form on $\mathbb{R}^n$ then there exists an orthogonal set of unit eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of the associated linear mapping l. If $y_1, \ldots, y_n$ are the coordinates of $\mathbf{x} \in \mathbb{R}^n$ with respect to the basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$, that is*

$$\mathbf{x} = y_1\mathbf{v}_1 + \cdots + y_n\mathbf{v}_n$$

then

$$q(\mathbf{x}) = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Corollary 18. *[1: p. 148] Then:*

- $\hookrightarrow$ *q is positive-definite if all the eigenvalues are positive*
- $\hookrightarrow$ *q is negative-definite if all the eigenvalues are negative, and*
- $\hookrightarrow$ *q is nondefinite if some are positive and some are negative*

Definition 22. [1: p. 149] Consider a quadratic form $q(\mathbf{x}) = \mathbf{x}^t A\mathbf{x}$ on $\mathbf{R}^n$ for which $|A| \neq 0$. Writing $A = (a_{ij})$ as usual, we denote by Δ_k the determinant of the upper left-hand $k \times k$ submatrix of A, that is

$$\Delta_k = \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix}$$

Thus $\Delta_1 = a_{11}$, $\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \ldots, \Delta_n = |A|$.

Theorem 19. *[1: p. 149] Let $q(\mathbf{x} = \mathbf{x}^t A\mathbf{x}$ be a quadratic form on $\mathbb{R}^n$ with $|A| \neq 0$. Then q is*

- $\hookrightarrow$ *positive-definite if and only if $\Delta_k > 0$ for each $k = 1, \ldots, n$*
- $\hookrightarrow$ *negative-definite if and only if $(-1)^k\Delta_k > 0$ for each $k = 1, \ldots, n$*
- $\hookrightarrow$ *nondefinite if neither of the previous two conditions is satisfied*

3 Successive approximations and implicit functions

Remark 8. [1: p. 160] Certain existence theorems in multivariable calculus deal with the possibility of solving equations or systems of equations. This section studies those theorems.

3.1 Newton's method and contraction mappings

Definition 23. [1: p. 160] Let $[a;b]$ be an interval on which $f'(x)$ is nonzero and $f(x)$ changes sign, so the equation $f(x) = 0$ has a single root $x_* \in [a;b]$. Given any arbitrary point $x_0 \in [a;b]$, a linear approximation gives the equation

$$f(x_0) - f(x_*) \approx f'(x_0)(x_0 - x_*)$$

Which rearranged gives

$$x_* = x_1 = x_0 - \frac{f(x_0)}{x'(x_0)}$$

Similarly we have the $(n+1)$st approximation from the nth approximation,

$$x_{n+1} = x_n - \frac{f(x_n)}{x'(x_n)}$$

This technique is called **Newton's method**.

Definition 24. [1: p. 162] The mapping $\phi : [a;b] \to [a;b]$ is called a **contraction mapping with contraction constant** $k < 1$ if

$$|\phi(x) - \phi(y)| \leq k|x - y|$$

for all $x, y \in [a;b]$.

Theorem 20. *[1: p. 162] Let $\phi : [a;b] \to [a;b]$ be a contraction mapping with contraction constant $k < 1$. Then ϕ has a unique fixed point x_*. Moreover, given $x_0 \in [a;b]$, the sequence $\{x_n\}_0^\infty$ defined inductively by $x_{n+1} = \phi(x_n)$ converges to x_*. In particular,*

$$|x_n - x_*| \leq \frac{k^n|x_0 - x_1|}{1 - k}$$

for each n.

Remark 9. Now we can dramatically simplify calculations of Newton's Method, using the following Theorem.

Theorem 21. *[1: p. 164] Let $f : [a; b] \to \mathbb{R}$ be a differentiable function with $f(a) < 0 < f(b)$ and $0 < m < f'(x) \leq M$ for $x \in [a; b]$. Given $x_0 \in [a; b]$, the sequence $\{x_n\}_0^\infty$ defined inductively by*

$$x_{n+1} = x_n - \frac{f(x_n)}{M}$$

converges to the unique root $x_* \in [a; b]$ of the equation $f(x) = 0$. In particular,

$$|x_n - x_*| \leq \frac{|f(x_0)|}{m}\left(1 - \frac{m}{M}\right)^n$$

for each n.

Remark 10. [1: p. 164] To finish what we proposed to do, we can turn the question on its head: given a point d_* where $f(x_*) = 0$, and a number y close to 0, can we find a point x near x_* such that $f(x) = y$?

Theorem 22. *[1: p. 164] Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathbb{C}^1$ function such that $f(a) = b$ and $f'(a) \neq 0$. Then there exist neighbourhoods $U = [a - \delta; a + \delta]$ of a and $V = [b - \epsilon; b + \epsilon]$ of b such that, given $y_* \in V$, the sequence $\{x_n\}_0^\infty$ defined inductively by $x_0 = a$ and*

$$x_{n+1} = x_n - \frac{f(x_n) - y_*}{f'(a)}$$

converges to a unique point $x_* \in U$ such that $f(x_*) = y_*$

4 Area and 1-dimensional integrals

4.1 Area

We develop a function $f : \mathbb{R}^n \to \mathbb{R}^+$ which captures the area of some set $A \subseteq \mathbb{R}^n$.

Theorem 23. *[2] Conditions for a well-defined area (later volume) $v(\cdot)$, $\forall A, B \subset \mathbb{R}^n$:*

- $\hookrightarrow$ *If $A \subseteq B$, then $v(A) \leq v(B)$*
- $\hookrightarrow$ *If $A \cap B \neq \emptyset$, then $v(A \cup B) = v(A) + v(B)$*
- $\hookrightarrow$ *If A is a translate of B, then $v(A) = v(B)$*

Example 1. [2] To define the area of a closed rectangle: for $I \in \mathbb{R}^2, I = [a, b] \times [c, d]$ (for a non-degenerate rectangle $I : a < b, c < d$). So $v(I) = (b - a)(d - c)$.

Definition 25. [2] Let S be a bounded subset of $\mathbb{R}^2$. We say its **area** is C if, given $\epsilon > 0$, there exists both:

- $\hookrightarrow$ A finite nonoverlapping collection of closed rectangles $\{I_k \subseteq S; k = 1, 2, \ldots, n\}$ with

$$\sum_{k=1}^{n} v(I_k) > C - \epsilon \iff C - \sum_{k=1}^{n} v(I_k) < \epsilon$$

- $\hookrightarrow$ A finite collection of closed rectangles $\{J_p; p = 1, 2, \ldots, m\}$ with $S \subseteq \cup_{p=1}^{m} J_p$ and

$$\sum_{p=1}^{m} v(J_p) < C + \epsilon \iff \sum_{p=1}^{m} v(J_p) - C < \epsilon$$

Example 2. [1: p. 205] Not every set has area. Consider

$$S = \{(x, y) \in \mathbb{R}^2 : x, y \in [0; 1]\}$$

with both x, y rational. Then S contains no nondegenerate rectangle. For condition 1, $I_k \subseteq S$, $V(I_k) = 0$, there is one single point in I_k. So $\sum v(I_k) = 0$. For condition 2, $S \subseteq \cup J_p$, so $\sum v(J_p) \geq v([0; 1 \times 0; 1]) = 1$. Finitely many because S is dense in $[0; 1] \times [0; 1]$.

Remark 11. To extend Definition 25 to $\mathbb{R}^n$ and not just $\mathbb{R}^2$, simply consider rectangular boxes instead of rectangles, and it becomes a definition of volume instead of area.

Definition 26. [1: p. 206] **Area** is a nonnegative values function $a : \mathcal{A} \to \mathbb{R}$, where $\mathcal{A}$ is the collection of all subsets of $\mathbb{R}^2$ which have area. It has the following properties:

- ↪ (A) If S and T have area and $S \subset T$, then $a(S) \leq a(T)$
- ↪ (B) If S and T are two nonoverlapping sets which have area, then so does $S \cup T$, and $a(S \cup T) = a(S) + a(T)$
- ↪ (C) If S and T are two congruent sets and S has area, then so does T, and $a(S) = a(T)$
- ↪ (D) If R is a rectangle, then $a(R)$ is the product of its base and height
- ↪ (E) If $S = \{(x, y) \in \mathbb{R}^2 : x \in [a; b] \text{ and } 0 \leq y \leq f(x)\}$ where f is a continuous nonnegative function on $[a; b]$, then S has area

4.2 Integrals in one dimension

Definition 27. [1: p. 206] Define the **integral** of a continuous function $f : [a; b] \to \mathbb{R}$. Suppose that f is nonnegative on $a; b]$, and consider the "ordinate set"

$$O_a^b(f) = \{(x, y) \in \mathbb{R}^2 : x \in [a; b], y \in [0; f(x)]\}$$

by property (E) of area, $O_a^b(f)$ has area. We definte the **integral** of the nonnegative continuous function $f : [a; b] \to \mathbb{R}$ by

$$\int_a^b f = a(O_a^b(f))$$

so the integral is exactly the area of the ordinate set of that function over that range.

Remark 12. [1: p. 207] Notice that, if $0 \leq m \leq f(x) \leq M$ on $[a; b]$, then we have $O_a^b(m) \subset O_a^b(f)$ and conversely $O_a^b(f) \subset O_a^b(M)$. So by property (A) of area,

$$m(b - a) \leq \int_a^b f \leq M(b - a)$$

Remark 13. [1: p. 207] Consider $c \in (a; b)$. Then $O_a^c(f)$ and $O_c^b(f)$ are nonoverlapping and have area, so

$$\int_a^b f = \int_a^c f + \int_a^b f$$

Definition 28. [1: p. 207] For f an arbitrary continuous function on $[a;b]$, we consider its **positive and negative parts** f^+ and f^-, defined on $[a;b]$ by

$$f^+(x) = \begin{cases} f(x) & f(x) \geq 0 \\ 0 & f(x) < 0 \end{cases}$$

And

$$f^-(x) = \begin{cases} -f(x) & f(x) \leq 0 \\ 0 & f(x) > 0 \end{cases}$$

so that $f = f^+ - f^-$, which are both nonnegative functions with continuity implied by the continuity of f. Therefore, the **integral** of f on $[a;b]$ can alternatively be defined as

$$\int_a^b f = \int_a^b f^+ - \int_a^b f^-$$

Theorem 24. *[1: p. 209] The Fundamental Theorem of Calculus: If $f : [a;b] \to \mathbb{R}$ is continuous, and $F : [a;b] \to \mathbb{R}$ is defined by $F(x) = \int_a^x f$, then F is differentiable, with $F' = f$.*

Remark 14. [1: p. 210] The Fundamental Theorem of Calculus gives us our usual method of computing integrals: if f is continuous and $G' = f$ on $[a;b]$, then

$$\int_a^b f = G(b) - G(a)$$

Theorem 25. *[1: p. 210] If the functions f and g are continuous on $[a;b]$ and $c \in \mathbb{R}$ then*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

And

$$\int_a^b (cf) = c\int_a^b f$$

Theorem 26. *[1: p. 211] If $f(x) \leq g(x)$ on $[a;b]$ then*

$$\int_a^b f \leq \int_a^b g$$

Notation 1. [1: p. 211] For clarity we often prefer to write

$$\int_a^b f = \int_a^b f(x)dx$$

Theorem 27. *[1: p. 211] Integration by substitution: Let f have a continuous derivative on $[a;b]$, and let g be continuous on $[c;d]$, where $f([a;b]) \subset [c;d]$. Then,*

$$\int_a^b g(f(x))f'(x)dx = \int_{f(a)}^{f(b)} g(u)du$$

Example 3. [1: p. 212] We will use integration by substitution to prove that a circle of radius r has area $A = \pi r^2$. π by definition is the area of the unit circle, so considering the arc that gives a circle in just one cartesian quadrant,

$$\frac{\pi}{4} = \int_0^1 (1-t^2)^{\frac{1}{2}} dt$$

so that

$$A = 4\int_0^r (r^2 - x^2)^{\frac{1}{2}} dx$$

$$A = 4r\int_0^r \left(1 - \frac{x^2}{r^2}\right)^{\frac{1}{2}} dx$$

Now apply the substitution $u = \frac{x}{r}$, so that $\frac{du}{dx} = \frac{1}{r}$, and the bounds now go from $x = 0 \implies \frac{x}{r} = 0$ up to $x = r \implies \frac{x}{r} = 1$. Then,

$$A = 4r^2\int_0^r \left(1 - \left(\frac{x}{r}\right)^2\right)^{\frac{1}{2}} D\left(\frac{x}{r}\right) dx$$

$$A = 4r^2\int_0^1 \left(1 - u^2\right)\frac{1}{2}du$$

Now we have normalized the circle of radius x so that we are simply integrating over one quadrant of the unity circle, which we already asserted is by definition

$$A = 4r^2\frac{\pi}{4}$$

$$A = \pi r^2$$

Theorem 28. *[1: p. 212] Integration by parts: If f and g are continuously differentiable on $[a; b]$, then*

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$$

Adopting the notation $u = f(x), v = g(x), du = f'(x)dx, dv = g'(x)dx$, then

$$\int udv = uv - \int vdu$$

5 Volume and n-dimensional integrals

5.1 Volume

Recall 2. [1: p. 214] A **closed interval** in $\mathbb{R}^n$ is a Cartesian product of closed intervals: it is any set $I = I_1 \times I_2 \times \cdots \times I_n$, where $I_j = [a_j; b_j] \subset \mathbb{R}^n$ with $j = 1, \ldots, n$. By definition, the volume of I is $v(I) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$.

Definition 29. [1: p. 214] Let A be a bounded subset of $\mathbb{R}^n$. Then we say that A is a **contented set** with volume $v(A)$ if, given $\epsilon > 0$, there exist both:

$\hookrightarrow$ (a) nonoverlapping closed intervals $I_1, \ldots, I_p \subset A$ such that $\sum_{i=1}^{p} v(I_i) > v(A) - \epsilon$

$\hookrightarrow$ (b) closed intervals $J_1, \ldots, J_q$ such that $A \subset \cup_{j=1}^{q} J_j$ and $\sum_{j=1}^{q} v(J_j) < v(A) + \epsilon$

Recall 3. [1: p. 215] We need a way of detecting which sets are contented. We'll do this with our idea of boundary. Recall that the boundary ∂A of the set $A \subset \mathbb{R}^n$ is the set of all those points of $\mathbb{R}^n$ that are limit points of both A and $\mathbb{R}^n \setminus A$.

Definition 30. [1: p. 215] The contented set A is called **negligible** if $v(A) = 0$. Referring to the definition of volume, this means that A is negligible iff, given $\epsilon > 0$, $\exists$ intervals $J_1, \ldots, J_q$ such that $A \subset \bigcup_{j=1}^{q} J_j$ and $\sum_{j=1}^{q} v(J_i) < \epsilon$ (since it must be under $\epsilon + 0$).

Theorem 29. *[1: p. 215] A union of finitely many neglibile sets is negligible. So is any subset of a negligible set.*

Theorem 30. *[1: p. 215] The bounded set A is contented if and only if its boundary is negligible.*

Corollary 31. *[1: p. 216] The intersection, union, or difference of two contented sets is contented.*

Remark 15. [1: p. 216] The above theorem is useful because negligible sets are often easy to spot.

Remark 16. [1: p. 216] The graph of a continuous function on a contented set is negligible.

Example 4. [1: p. 216] Let B^n be the unit ball in $\mathbb{R}^n$, with $\partial B^n = S^{n-1}$, the unit sphere of dimension $n - 1$. Assume (by induction) that B^{n-1} is a contented subset of $\mathbb{R}^{n-1} \subset \mathbb{R}^n$. Then S^{n-1} is the union of the graphs of the continuous functions

$$f^+(x_1, \ldots, x_{n-1}) = \left[\sum_{i=1}^{n-1} x_i^2\right]^{\frac{1}{2}}$$

and

$$f^-(x_1, \ldots, x_{n-1}) = -\left[\sum_{i=1}^{n-1} x_i^2\right]^{\frac{1}{2}}$$

By Remark 16, since S^{n-1} is the union of the graphs of continuous functions on a negligible set, it is a negligible subset of $\mathbb{R}^n$. Since $S^{n-1} = \partial B^n$, B^n is negligible by Theorem 30.

5.2 Integrals in n dimensions

Definition 31. [1: p. 216] Given a nonnegative function $f : \mathbb{R}^n \to \mathbb{R}$, consider the **ordinate set**

$$O_f = \{(x_1, \ldots, x_{n+1}\} \in \mathbb{R}^{n+1} : 0 < x_{n+1} \leq f(x_1, \ldots, x_n)\}$$

We want the ordinate set to be bounded, for which we require that f is bounded and has bounded support. $f : \mathbb{R}^n \to \mathbb{R}$ is **bounded** if there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}^n$, and **has bounded support** if there exists a closed interval $I \subset \mathbb{R}^n$ such that $f(x) = 0$ if $x \notin I$.

Definition 32. [1: p. 217] Given $f : \mathbb{R}^n \to \mathbb{R}$, we define the **positive part** and **negative part** of f as

$$f^+(x) = \max(0, f(x))$$

And

$$f^-(x) = \max(0, -f(x))$$

Notice $f = f^+ - f^-$.

Definition 33. [1: p. 217] Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is bounded and has bounded support. We then say that f is **Riemann integrable** iff the ordinate sets O_{f^+} and O_{f^-} are both contented, in which case

$$\int f \equiv v(O_{f^+}) - v(O_{f^-})$$

Remark 17. [1: p. 217] Under Definition 33, $\int f$ is the volume of the region in $\mathbb{R}^{n+1}$ above $\mathbb{R}^n$ and below the graph of f, minus the volume of the region below $\mathbb{R}^n$ and above the graph of f. This is just like the 1 dimensional case we studied in the last section.

Definition 34. [1: p. 217] The **characteristic function** ϕ_A of f over some set A is

$$\phi_A(x) = \begin{cases} 1 & x \in A \\ 0 & \text{else} \end{cases}$$

Definition 35. [1: p. 217] Provided that $f\phi_A$ is integrable, we can then define

$$\int_A f = \int f\phi_A$$

Definition 36. [1: p. 218] The following four conditions are "axioms" of integration (though given the information we have already claimed, by now they are really just theorems):

- ↪ (I) The set $\mathcal{F}$ of integrable functions is a vector space
- ↪ (II) The mapping $\int : \mathcal{F} \to \mathbb{R}$ is linear
- ↪ (III) If $f \geq 0$ everywhere, then $\int f \geq 0$
- ↪ (IV) If the set A is contented, then $\int \phi_A = v(A)$

Definition 37. [1: p. 219] A function f is called **admissible** if

- ↪ f is bounded
- ↪ f has bounded support
- ↪ f is continuous except on a negligible set

Theorem 32. *[1: p. 219] Every admissible function is integrable.*

Theorem 33. *[1: p. 221] If f is admissible and A is contented, then $f\phi_A$ is admissible (and therefore integrable).*

Theorem 34. *[1: p. 221] If f and g are admissible functions on $\mathbb{R}^n$ with $f \leq g$ everywhere, and A is a contented set, then*

$$\int_A f \leq \int_A g$$

Theorem 35. *[1: p. 221] If the admissible function f satisfies $|f(x)| \leq M$ for all $x \in \mathbb{R}^n$, and A is contented, then*

$$\left| \int_A f \right| \leq M v(A)$$

Theorem 36. *[1: p. 222] If A and B are contented sets with $A \cap B$ negligible, and f is admissible, then*

$$\int_{A \cup B} = \int_A f + \int_B f$$

Theorem 37. *[1: p. 222] Let A be contented, and suppose the the admissible functions f and g agree except on the negligible set D. Then*

$$\int_A f = \int_A g$$

6 Step functions and Reimann sums

6.1 Step functions

Definition 38. [1: p. 223] The function $h : \mathbb{R}^n \to \mathbb{R}$ is called a **step function** if h can be written as a linear combination of characteristic functions $\phi_1, \ldots, \phi_p$ of (not necessarily closed) intervals $I_1, \ldots, I_p$ whose interiors are mutually disjoint, that is

$$h = \sum_{i=1}^{p} a_i \phi_i$$

with coefficients $a_i \in \mathbb{R}$.

Remark 18. So, can we sum weighted characteristic functions to get h? It's a bit of a sideways take on the idea of a characteristic function, which maps any $x \in A$ onto 1. But if we can reweight to build h, it's a step function.

Definition 39. [1: p. 223] If h is a step function, with $h = \sum_{i=1}^{p} a_i \phi_i$ as above, then h is integrable, with

$$\int h = \sum_{i=1}^{p} a_i v_i(I_i)$$

Theorem 38. *[1: p. 224] If h is a step function, then $\int ch = c \int h$.*

Theorem 39. *[1: p. 224] If h and k are step functions, then so is $h + k$, and $\int (h + k) = \int h + \int k$.*

Theorem 40. *[1: p. 225] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function with bounded support. Then f is integrable if, given $\epsilon > 0$, there exist step functions h and k such that*

$$h \leq f \leq k$$

And

$$\int (k - h) < \epsilon$$

in which case

$$\int h \leq \int f \leq \int k$$

Example 5. [1: p. 227] Let's use the step function theorem to prove that all continuous functions are integrable. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function with compact support. Since it follows that the nonnegative functions f^+ and f^- are continuous, for simplicity let's assume that f nonnegative. let $Q \subset \mathbb{R}^n$ be a close interval such that $f = 0$ outside Q. Then f is uniformly continuous on Q, so given $\epsilon > 0$, $\exists \delta > 0$ so

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{v(Q)}$$

Then let m_i be the minimum value of f on Q, M_i the max of f on Q, ϕ_i the chatacteristic function of $\text{int}(Q)$, and ψ_i the characteristic function of Q_i, and $h = \sum_{i=1}^{l} m_i \phi_i$ and $k = \sum_{i=1}^{l} M_i \psi_i$, then

$$\int (k - h) = \sum_{i=1}^{l} (M_i - m_i) v(Q_i) < \frac{\epsilon}{v(Q)} \sum_{i=1}^{l} v(Q_i) < \epsilon$$

Then Theorem 40 guarantees integrability.

6.2 Riemann sums

Definition 40. [1: p. 228] A **partition** of the interval Q is a collection $\mathcal{P} = \{Q_1, \ldots, Q_k\}$ of closed intervals with disjoint interiors such that $Q = \cup_{i=1}^{k} Q_i$.

Definition 41. [1: p. 228] A **mesh** of partition $\mathcal{P}$ is the maximum of the diameters of the Q_i in the partition, recalling that the diameter of a set D is $\text{diam}(D) = \sup\{d(x, y) : \forall x, y \in D\}$

Definition 42. [1: p. 229] A **selection** for partition $\mathcal{P}$ is a set $\mathcal{L} = \{x_1, \ldots, x_k\}$ of points such that $x_i \in Q_i$ for each i.

Theorem 41. *[1: p. 229] If $f : \mathbb{R}^n \to \mathbb{R}$ is a function such that $f = 0$ outside of Q, then the Riemann sum for f corresponding to the $\mathcal{P}$ and selection $\mathcal{L}$ is*

$$R(f, \mathcal{P}, \mathcal{L}) = \sum_{i=1}^{k} f(x_i) v(Q_i)$$

which is just the integral of the step function h where $h = \sum_{i=1}^{k} f(x_i)\phi_i$, with ϕ_i the characteristic function of Q_i.

Theorem 42. *[1: p. 229] Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is bounded and vanishes outside the interval Q. Then f is integrable with $\int f = I$ if, given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|I - R(f, \mathcal{P}, \mathcal{L})| < \epsilon$$

whenever $\mathcal{P}$ is a partition of Q with mesh $< \delta$ and $\mathcal{L}$ is a selection for $\mathcal{P}$.

Remark 19. [1: p. 231] So the operation of integration is a limit process! Hot dog.

Theorem 43. *[1: p. 231] If $f : \mathbb{R}^n \to \mathbb{R}$ is an integrable function which vanishes outside the interval Q, and $\{\mathcal{P}\}_1^\infty$ is a sequence of partitions of Q with associated selections $\{\mathcal{L}_k\}_1^\infty$ such that $\lim_{k\to\infty}(\text{mesh of } \mathcal{P}_k) = 0$, then*

$$\int f = \lim_{k\to\infty} R(f, \mathcal{P}_k, \mathcal{L}_k)$$

Example 6. [1: p. 231] Let $A \in \mathbb{R}^m$ and $B \in \mathbb{R}^n$ be contented sets, and $f : A \times B \to \mathbb{R}$ a continuous function. Define $g : A \to \mathbb{R}$ by

$$g(x) = \int_B f(x, y)dy = \int_B f_x$$

Where $f_x(y) = f(x, y)$. Is g continuous on A? This is the same as asking

$$\lim_{x\to a} \int_B f(x, y)dy = \int_B \lim_{x\to a} f(x, y)dy$$

for each $a \in A$. This is true if f is uniformly continuous.

Example 7. [1: p. 232] Let $\{f_n\}_1^\infty$ be a sequence of integrable functions on the contented set A, which converges pointwise to the integrable function $f : A \to \mathbb{R}$. In a homework exercise we show that

$$\lim_{n\to\infty} \int_A f_n = \int \lim_{n\to\infty} f_n = \int f$$

Example 8. [1: p. 232] Let $f : A \times J \to \mathbb{R}$ be a continuous function, where $A \subset \mathbb{R}^n$ is contented and $J \subset \mathbb{R}$ is an open interval. Define the partial derivative $D_2 f : A \times J \to \mathbb{R}$ by

$$D_2 f(x, t) = \lim_{h\to 0} \frac{f(x, t + h) - f(x, t)}{h}$$

and the function $g : J \to \mathbb{R}$ by $g(t) = \int_A f(x, t)dx$. Then, if $D_2 f$ is uniformly continuous on $A \times J$,

$$g'(t) = \int_A D_2 f(x, t)dx$$

So

$$\frac{\partial}{\partial t}\int_A f(x,t)dx = \int_A \frac{\partial}{\partial t} f(x,t)dx$$

7 Iterated integrals, Fubini's theorem, and Cavalieri's principle

Theorem 44. *[1: p. 235] If f is a continuous function on the rectangle $R = [a;b] \times [c;d] \in \mathbb{R}^2$, then*

$$\int_a^b \left(\int_c^d f(x,y)dy \right) dx = \int_c^d \left(\int_a^b f(x,y)dx \right) dy$$

Remark 20. [1: p. 235] The main theorem of this section will show that $\int_R$ is also equal to the value above.

Theorem 45. *[1: p. 238] Fubini's Theorem: Let $f : \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be an integrable function such that, for each $x \in \mathbb{R}^m$, the function $f_x : \mathbb{R}^n \to \mathbb{R}$, defined by $f_x(y) = f(x,y)$, is integrable. Given contented sets $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$, let $F : \mathbb{R}^m \to \mathbb{R}$ be defined by*

$$F(x) = \int_B f_x = \int_B f(x,y)dy$$

Then F is integrable, and

$$\int_{A \times B} f = \int_A F$$

Or, in the previous notation style,

$$\int_{A \times B} = \int_A \left(\int_B f(x,y)dy \right) fx = \int_B \left(\int_A f(x,y)dx \right) dy$$

Theorem 46. *[1: p. 240] Cavalieri's Principle: Let A be a contented subset of $\mathbb{R}^{n+1}$, with $A \subset R \times [a;b]$, where $R \subset \mathbb{R}^n$ and $[a;b] \subset \mathbb{R}$ are intervals.*

$$A_t = \{x \in \mathbb{R}^n : (x,t) \in A\} \subset \mathbb{R}^n$$

is contented for each $t \in [a;b]$, and write $A(t) = v(A_t)$. Then

$$v(A) = \int_a^b A(t)dt$$

Remark 21. [1: p. 241] Typically Cavalieri's Principle is used to compute volumes of revolution in $\mathbb{R}^3$, in which a solid is revolved around an axis or point.

Theorem 47. *[1: p. 241] Let $f : [a; b] \to \mathbb{R}$ be a positive continuous function. Denote by A the set in $\mathbb{R}^3$ obtained by revolving about the x-axis the ordinate set of f. Then $A(t) = \pi[f(t)]^2$, so by Theorem 46,*

$$v(A) = \pi \int_a^b [f(x)]^2 dx$$

Example 9. [1: p. 241] To compute the volume of the 3-dimensional ball B_r^3 of radius r, take $f(x) = (r^2 - x^2)^{\frac{1}{2}}$ on $[-r; r]$. Then we obtain

$$v(B_r^3) = \pi \int_{-r}^{r} (r^2 - x^2) dx$$

$$v(B_r^3) = \pi r^2 \int_{-r}^{r} 1 dx - \pi \int_{-r}^{r} x^2 dx$$

$$v(B_r^3) = \pi r^2 (r - (-r)) - \pi \frac{1}{3}(r^3 - (-r)^3)$$

$$v(B_r^3) = 2\pi r^3 - \frac{2}{3}\pi r^3$$

$$v(B_r^3) = \frac{4}{3}\pi r^3 dx$$

Theorem 48. *[1: p. 242] If $A \subset \mathbb{R}^n$ is a contented set and f_1 and f_2 are continous functions on A so that $f_1 \leq f_2$, then*

$$C = \{(x, y) \in \mathbb{R}^{n+1} : x \in A \text{ and } f_1(x) \leq x \leq f_2(x)\}$$

is a contented set. If $g : C \to \mathbb{R}$ is continuous, then

$$\int_C g = \int_A \left(\int_{f_1(x)}^{f_2(x)} g(x, y) dy \right) dx$$

Example 10. [1: p. 242] If g is a continuous function on the unit ball $B^3 \subset \mathbb{R}^3$, then

$$\int_{B^3} g = \int_{B^2} \left(\int_{-(1-x^2-y^2)^{\frac{1}{2}}}^{+(1-x^2-y^2)^{\frac{1}{2}}} g(x, z) dz \right) dx$$

where $x = (w, y) \in B^2$

Theorem 49. *[1: p. 242] The formula we promised at the start of the section is obtained by setting $m = n = 1$ in Fubini's theorem, Theorem 45, with $A = [a; b]$ and $B = [c; d]$, yielding*

$$\int_{A\times B} f = \int_a^b \left(\int_c^d f(x, y) dy \right) dx$$

8 Change of variables

Remark 22. [1: p. 245] At the heart of the change of variables idea is a really lovely trick. The idea is that arbitrarily ugly regions can (under certain conditions) be understood as *rectangles mapped onto arbitrary ugly regions* in a way that preserves their area, so that by identifying that rectangle and computing its area we can get the area of a shape that is not amenable to direct computation by Fubini's Theorem.

The basic problem is as follows: $T : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping which is both $\mathcal{C}^1$ and is $\mathcal{C}^1$-invertible, meaning that it is one-to-one and that its inverse $T^{-1} : T(U) \to U$ is also $\mathcal{C}^1$. Given an integrable function $f : T(A) \to \mathbb{R}$, we want to transform the integral $\int_{T(A)} f$ into an appropriate integral over A.

Theorem 50. *[1: p. 246] If $\lambda : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping, and $B \subset \mathbb{R}^n$ is contented, then $\lambda(B)$ is also contented, and*

$$v(\lambda(B)) = |det\ \lambda| v(B)$$

Corollary 51. *[1: p. 248] If $A \subset \mathbb{R}^n$ is a contented set and $\rho : \mathbb{R}^n \to \mathbb{R}^n$ is a rigid motion, then $\rho(A)$ is contented and $v(\rho(A)) = v(A)$.*

Remark 23. [1: p. 248] So, rigid motions preserve contentedness and volume.

Example 11. [1: p. 248] Consider the ball

$$B_r^n = \{x \in \mathbb{R}^n : \sum_1^n x_i^2 \leq r^2$$

of radius r in $\mathbb{R}^n$. Note that B_r^n is the image of the unit n-dimensional ball $B_1^n \subset \mathbb{R}^n$ under the linear mapping $T(x) = rx$. Since $|\det T'| = |\det\ | = r^n$, by Theorem 50,

$$v(B_r^n) = \alpha_n r^n$$

so the volume of an n-dimensional ball is proportional to the nth power of its radius. Let α_n denote the volume of the unit n-dimensional ball B_1^n, so

$$v(B_r^n) = \alpha_n r^n$$

Theorem 52. *[1: p. 250] Let Q be an interval centered at the point $a \in \mathbb{R}^n$, and suppose $T : U \to \mathbb{R}^n$ is a $\mathcal{C}^1$-invertible mapping on a neighbourhood U of Q. If there exists $\epsilon \in (0; 1$ such that*

$$||dT_a^{-1} \circ dT_x - I|| \leq \epsilon$$

For all $X \in Q$, then $T(Q)$ is contented with

$$(1-\epsilon)^n |det\, T'(a)| v(Q) \leq v(T(Q)) \leq (1+\epsilon)^n |det\, T'(a)| v(Q)$$

Remark 24. [1: p. 250] So $T(Q)$ is some messy function that could be arbitrarily squiggly. dT is the linear approximation of $T(Q)$, and it plays the important role of boxing in T so that it can get arbitrarily close to being an interval. With this approximation we can get $v(T(Q))$. $(1+\epsilon)^n$ can be exceptionally small, so the idea is that if dT_x is approximately equal to dT_a for all $x \in Q$, which should be true for sufficiently small Q, then $v(T(Q))$ is approximately equal to $|\det\, T'(a)| v(Q)$. So $|\det\, T'(a)|$ is a sort of local magnification factor.

Theorem 53. *[1: p. 252] Change of variables formula: Let Q be an interval in $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping which is $\mathcal{C}^1$-invertible on a neighbourhood of Q. If $f : \mathbb{R}^n \to \mathbb{R}$ is an integrable function such that $f \circ T$ is also integrable, then*

$$\int_{T(Q)} f = \int_Q (f \circ T) |det\, T'|$$

Example 12. [1: p. 257] Let A denote the annular region in the plane bounded by the circles of radii a and b centered at the origin, with $a < b$. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the polar coordinates mapping defined by

$$T(r, \theta) = (r\cos\theta, r\sin\theta)$$

Then A is the image under T of the rectangle

$$Q = \{(r, \theta) \in \mathbb{R}^2 : r \in [a; b] \text{ and } \theta \in [0; 2\pi]\}$$

T is not one to one on Q, because eg $T(r, 0) = T(r, 2\pi)$, but it is $\mathcal{C}^1$-intervible on the interior of Q which is enough. Then the integral over some integrable function f is

$$\int_A f = \int_Q (f \circ T) |\det\, T'|$$

To calculate $|\det\, T'|$,

$$|\det\, T'| = \begin{vmatrix} \frac{\partial T_1}{\partial r} & \frac{\partial T_1}{\partial \theta} \\ \\ \frac{\partial T_2}{\partial r} & \frac{\partial T_2}{\partial \theta} \end{vmatrix}$$

$$|\det T'| = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix}$$

$$|\det T'| = r\cos^2(\theta) + r\sin^2(\theta)$$

$$|\det T'| = r$$

Then,

$$\int_A f = \int_0^{2\pi} \int_a^b f(r\cos(\theta), r\sin(\theta))r \; dr \; d\theta$$

Example 13. [1: p. 256] Let A be an ice cream cone shape bounded above by the sphere of radius 1 and below by a cone with vertex angle $\frac{\pi}{6}$. If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the spherical coordinates mapping defined by

$$T(\rho, \phi, \theta) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))$$

Note that we're given the "vertex angle" of the cone, which is the angle from one edge of the cone to the "opposite" edge. We care not about that, but about the angle from the axis that it's centered on to any edge of the cone: this is the angle that defines how much of the cone intersects with the circle that is also bounding the object of interest in each quadrant. So integrating over radius 1 and 2π radians gives us how much of the (call it) z direction is filled up by the cone; then integrating that amount again over $\frac{1}{2}\frac{\pi}{6} = \frac{\pi}{12}$ gives us how much of the x and y dimensions are filled up by the figure.

Then, A is the image under T of the interval

$$Q = \{(\rho, \phi, \theta) : \rho \in [0;1], \phi \in [0; \frac{\pi}{12}], \theta \in [0; 2\pi]\}$$

Now we seek $|\det T'|$:

$$|\det T'| = \begin{vmatrix} \frac{\partial f_1}{\partial \rho} & \frac{\partial f_1}{\partial \phi} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial \rho} & \frac{\partial f_2}{\partial \phi} & \frac{\partial f_3}{\partial \theta} \\ \frac{\partial f_3}{\partial \rho} & \frac{\partial f_3}{\partial \phi} & \frac{\partial f_3}{\partial \theta} \end{vmatrix}$$

$$|\det T'| = \begin{vmatrix} \sin(\phi)\cos(\theta) & \rho\cos(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & \rho\cos(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) \\ \cos(\phi) & -\rho\sin(\phi) & 0 \end{vmatrix}$$

$$|\det T'| = \sin(\phi)\cos(\theta)\Big(-\rho\sin(\phi)\cos(\theta)\Big)\Big(-\rho\sin(\phi)\Big) - \rho\cos(\phi)\cos(\theta)\Big(-\rho\sin(\phi)\cos(\theta)\cos(\phi)\Big) + \Big(-\rho\sin(\phi)\sin(\theta)\Big)\Big(\big(\sin(\phi)\sin(\theta)\big)\big(-\rho\sin(\phi)\big) - \rho\cos(\phi)\sin(\theta)\cos(\phi)\Big)$$

$$|\det T'| = \rho^2\sin^3(\phi)\big(\sin^2(\theta)+\cos^2(\theta)\big) + \rho^2\cos^2(\phi)\sin(\phi)\big(\cos^2(\theta)+\sin^2(\theta)\big)$$

Then with two applications of $\sin^2(\theta)+\cos^2(\theta)=1$,

$$|\det T'| = \rho^2\sin^3(\phi) + \rho^2\cos^2(\phi)\sin(\phi)$$

$$|\det T'| = \rho^2\sin(\phi)\big(\sin^2(\phi)+\cos^2(\phi)\big)$$

$$|\det T'| = \rho^2\sin(\phi)$$

Noting that T is $\mathcal{C}^1$-invertible on the interior of Q, apply Cavalieri's Principle and Fubini's Theorem for:

$$\int_A = \int_0^{2\pi}\int_0^{\frac{\pi}{12}}\int_0^1 f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta$$

Example 14. [1: p. 257] To compute the volume of a unit ball $B^3 \subset \mathbb{R}^3$, take $f(x)=1$, and T is the sperical coordinates mapping. Then notice that if we were to integrate up to radius 1 and ϕ from 0 to 2π, then we have identified the volume of a semicircle above the third axis: to get the full volume, we need to integrate that semicircle's area over all 2π degrees of the third axis to fill up a 3D circle. By this informal argument,

$$Q = \{(\rho,\phi,\theta) : \rho\in[0;1], \phi\in[0;\pi], \theta\in[0;2\pi]\}$$

then $T(Q)=B^3$, and T is $\mathcal{C}^1$-invertible on the interior of Q, so

$$v(B) = \int_{B^3} 1$$

$$v(B) = \int_Q (f\circ T)|\det T'|$$

$$v(B) = \int_0^1\int_0^{\pi}\int_0^{2\pi} 1\cdot\rho^2\sin(\phi)\ d\theta\ d\phi\ d\rho$$

$$v(B) = \int_0^1\int_0^{\pi}\Big[\rho^2\sin(\phi)\theta\Big]_{\theta=0}^{\theta=2\pi}\ d\phi\ d\rho$$

$$v(B) = 2\pi \int_0^1 \int_0^\pi \rho^2 \sin(\phi) \; d\phi \; d\rho$$

$$v(B) = 2\pi \int_0^1 \Big[\rho^2(-\cos(\phi))\Big]_{\phi=0}^{\phi=\pi} \; d\rho$$

$$v(B) = 2\pi \int_0^1 (1)\rho^2 - (-1)\rho^2 \; d\rho$$

$$v(B) = 2\pi \int_0^1 2\rho^2 \; d\rho$$

$$v(B) = 2\pi \left[\frac{2}{3}\rho^3\right]_{\rho=0}^{\rho=1}$$

$$v(B) = 2\pi\left(\frac{2}{3}\right)$$

$$v(B) = \frac{4}{3}\pi$$

Example 15. [1: p. 258] Suppose we want to compute

$$\int\int_R (\sqrt{x} + \sqrt{y})^{\frac{1}{2}} \; dx \; dy$$

where R is the region in the first quadrant bounded by the coordinate axes and the parabolic curve $\sqrt{x} + \sqrt{y} = 1$. Let's think about how to choose an appropriate T. The substitution $u = \sqrt{x}$ and $v = \sqrt{y}$ looks like it simplifies things, so consider the mapping $T(u, v) = (u^2, v^2)$. This produces a triangle bounded by the axes and the line $u + v = 1$. Then $\det T' = 4uv$, so by the change of variables formula

$$\int\int_R (\sqrt{x} + \sqrt{y})^{\frac{1}{2}} \; dx \; dy = 4 \int\int_Q uv(u + v)^{\frac{1}{2}} \; du \; dv$$

$$\int\int_R (\sqrt{x} + \sqrt{y})^{\frac{1}{2}} \; dx \; dy = 4 \int_0^1 \left(\int_0^{1-v} (u + v)^{\frac{1}{2}} uv\right) dv$$

Then we can integrate by substitution.

9 Improper integrals and absolutely integrable functions

Remark 25. [1: p. 268] Up to this point we have built our notion of volumes and integration entirely on bounded sets, because ordinate sets are necessarily bounded. But we might very naturally want to extend our notion of volume to be able to cover unbounded sets like, for example, the "area" under the curve $y = x^2$. To integrate a function $f : U \to \mathbb{R}$, with at least one of f and U unbounded, the idea will be to choose a sequence $\{A_n\}_{n=1}^{\infty}$ of subsets which "fill up" U, each of which f is integrable on U, so that

$$\int_u f = \lim_{n\to\infty} \int_{A_n} f$$

provided that this limit exists. It's also important that our definition ensures that $\int_u f$ will be independent of the particular sequence $\{A_n\}_{n=1}^{\infty}$ that is chosen.

Definition 43. [1: p. 270] A function $f : U \to \mathbb{R}$ is **locally integrable** on U if f is integrable on every compact contented subset of U.

Example 16. [1: p. 270] $f(x) = \frac{1}{x^2}$ is not integrable on $(1; \infty)$, but it is locally integrable there.

Example 17. [1: p. 270] $f(x) = \frac{1}{\sqrt{x}}$ is not integrable on $(0; 1)$, but it is locally integrable there.

Definition 44. [1: p. 270] Let $f : U \to \mathbb{R}$ be a locally integrable function on the open set $U \subset \mathbb{R}^n$. Then f is **absolutely integrable** on U if, given $\epsilon > 0$, there is a compact contented subset B_ϵ of U such that

$$\left| \int_A f - \int_{B_\epsilon} f \right| < \epsilon$$

for every compact contented set $A \subset U$ which contains B_ϵ.

Theorem 54. *[1: p. 271] If the function $f : U \to \mathbb{R}$ is absolutely integrable, then so is its absolute value $|f|$.*

Definition 45. [1: p. 271] By I denote the **improper Riemann integral** of f on the open set $U \subset \mathbb{R}^n$, and temporarily denote it $I_U f$ until we verify that $I_U f = \int_U f$ whenever f integrable on U.

Definition 46. [1: p. 272] The sequence $\{A_k\}_1^\infty$ of compact contented subsets of U is called an **approximating sequence** for U if both of the following hold:

↪ (a) $A_k \subset A_{k+1}$ for each $k \geq 1$,

↪ (b) $U = \cup_{k=1}^\infty \text{ int } A_k$

Theorem 55. *[1: p. 272] Suppose that f is absolutely integrable on the open set $U \subset \mathbb{R}^n$. Then there exists a number $I = I_U f$ with the property that, given $\epsilon > 0$, there exists a compact contented subset C_ϵ such that*

$$\left| I - \int_A f \right| < \epsilon$$

for every compact contented subset A of U containing C_ϵ. Furthermore,

$$I_U f = \lim_{k \to \infty} \int_{A_k} f$$

for every approximating sequence $\{A_k\}_1^\infty$ for U.

Theorem 56. *[1: p. 273] If $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}$ is integrable, then f is absolutely integrable, and*

$$I_U f = \int_U f$$

Definition 47. [1: p. 273] The improper integral is a **bounded improper integral** if there exists $M > 0$ such that

$$\left| \int_A f \right| \leq M$$

Remark 26. [1: p. 273] We have a much easier test for absolute integrability of *nonnegative* functions:

Theorem 57. *[1: p. 273] Suppose that a nonnegative function f is locally integrable on the open set U. Then f is absolutely integrable on U if and only if $\int_U f$ is bounded, in which case $\int_U f$ is the least upper bound of the values $\int_A f$, for all compact contented sets $A \subset U$.*

Corollary 58. *[1: p. 274] Suppose that the nonnegative function f is locally integrable on the open set U, and let $\{A_k\}_1^\infty$ be an approximating sequence for U. Then f is absolutely integrable with*

$$\int_U f = \lim_{k\to\infty} \int_{A_k} f$$

provided that this limit exists and is finite.

Example 18. [1: p. 274] Let A denote the solid annular region $B_b^n \setminus \text{ int } B_a^n \subset \mathbb{R}^n$, and let f be any spherically symmetric continuous function on A. That is,

$$f \circ T(\rho, \phi_1, \ldots, \phi_{n-2}, \theta) = g(\rho)$$

for some function g, where $T : \mathbb{R}^n \to \mathbb{R}^n$ is the n-dimensional spherical coordinates mapping. By change of variables,

$$\int_A f = \int_a^b \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} g(\rho)\rho^{n-1} \sin^{n-2}(\phi_1) \cdots \sin(\phi_{n-2})\ d\theta\ d\phi_1 \cdots d\phi_{n-2}\ d\rho$$

$$\int_A f = \sigma_n \int_a^b g(\rho)\rho^{n-1}\ d\rho$$

where

$$\sigma = \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \sin^{n-2}(\phi_1) \cdots \sin(\phi_{n-2})\ d\theta\ d\phi_1 \cdots d\phi_{n-2}$$

Example 19. [1: p. 275] Let $U = \mathbb{R}^n - B_1^n$, and $f(x) = \frac{1}{|x|^p}$ where $p > n$. Writing $A_k = B_k^n \setminus B_1^n$, by Corollary 58

$$\int_U f = \lim_{k\to\infty} \int_{A_k} f$$

$$\int_U f = \lim_{k\to\infty} \sigma_n \int_1^k \rho^{n-p-1}\ d\rho$$

$$\int_U f = \lim_{k\to\infty} \frac{\sigma_n}{n-p}(k^{n-p} - 1)$$

Then with $k^{n-p} \to 0$ since $n - p < 0$,

$$\int_U f = \frac{\sigma_n}{p-n}$$

Example 20. [1: p. 275] Let U denote the interior of the unit ball with the origin deleted, and $f(x) = \frac{1}{|x|^p}$ with $p < n$. Now the function f is unbounded, rather than the set U. Writing $A_k = B_1^n \setminus B_{\frac{1}{k}}^n$, by Corollary 58

$$\int_U f = \lim_{k\to\infty} \int_{A_k} f$$

$$\int_U f = \lim_{k\to\infty} \sigma_n \int_{\frac{1}{k}}^{1} \rho^{n-p-1}\, d\rho$$

$$\int_U f = \lim_{k\to\infty} \frac{\sigma_n}{n-p}\left[1 - \left(\frac{1}{k}\right)^{n-p}\right]$$

Then because $p < n$,

$$\int_U f = \frac{\rho_n}{n-p}$$

Remark 27. [1: p. 276] We use Corollary 58 as a working definition for nonnegative locally integrable functions: if the improper integral $\int_U f$ exists, in which case f is absolutely integrable over U, then we can choose any appropriate approximating sequence $\{A_k\}_1^\infty$ for U, and if $\lim_{k\to\infty} \int_{A_k} f$ is finite, then the function must have been absolutely integrable and in this case the value of $\int_U f$ is whatever limit we obtained for the integral of the approximating sequence.

In the case of a function that is not necessarily nonnegative, then we have to know in advance that it was absolutely integrable. In this case the easiest way is to use a comparison test.

Corollary 59. *[1: p. 276] Let $f : U \to \mathbb{R}$ be locally integrable. If $|f|$ is absolutely integrable on U, then so is f.*

Corollary 60. *[1: p. 276] Comparison Test: Suppose that f and g are locally integrable on U with $0 \le f \le g$ implies immediately that $\int_U f$ is also bounded, so f is absolutely integrable*

Example 21. [1: p. 276] Because we know that $\frac{1}{x^2}$ is absolutely integrable on $U = (1;\infty)$ by previous examples, the following functions are all absolutely integrable on U:

$$f(x) = \frac{1}{x^2+1}$$

$$f(x) = \frac{1}{x^3+1}$$

$$f(x) = e^{-x}$$

$$f(x) = \frac{\sin(x)}{x^2}$$

This last one requires an application of the absolute value implication, Corollary 59. Similarly, the following functions are absolutely integrable on $U = (0;1)$ by comparison with the function $g(x) = \frac{1}{x^{\frac{1}{2}}}$.

$$f(x) = \frac{1}{(x+1)^{\frac{1}{2}}}$$

$$f(x) = \frac{\cos(x)}{x^{\frac{1}{2}}}$$

Remark 28. [1: p. 277] Suppose that f is locally integrable on the open interval $(a;b)$, where it is permitted for $a = -\infty$ or $b = \infty$ or both. We want to define the improper integral

$$\int_a^b f(x)\ dx$$

in contrast with the improper integral $\int_{(a;b)} f(x)\ dx$ which exists if f is absolutely integrable on $(a;b)$. We will see that $\int_a^b f(x)\ dx$ exists in some cases where f is not absolutely integrable on $(a;b)$, but that when both exist

$$\int_a^b f(x)\ dx = \int_{(a;b)} f(x)\ dx$$

Remark 29. [1: p. 277] We obtain this new improper integral by considering four special approximating sequences for $(a;b)$:

$$\int_a^b f = \lim_{n\to\infty} \int_{a+\frac{1}{n}}^{b-\frac{1}{n}} f$$

The above integral slightly undershoots a finite a from above (landing definitely in a place where f is still defined) and slightly undershoots a finite b from below. The other options, if one of the bounds is infinite, are:

$$\int_a^\infty f = \lim_{n\to\infty} \int_{a+\frac{1}{n}}^{n} f$$

$$\int_{-\infty}^\infty f = \lim_{n\to\infty} \int_{-n}^{n} f$$

$$\int_{-\infty}^b f = \lim_{n\to\infty} \int_{-n}^{b-\frac{1}{n}} f$$

When the appropriate limit exists, we say that the integral converges. Otherwise, it diverges.

Example 22. [1: p. 277] The integral

$$\int_{-\infty}^{\infty} \left[\frac{1+x}{1+x^2}\right] dx$$

converges, because:

$$\int_{-\infty}^{\infty} \left[\frac{1+x}{1+x^2}\right] dx = \lim_{n\to\infty} \int_{-n}^{n} \frac{1+x}{1+x^2}\ dx$$

$$\int_{-\infty}^{\infty} \left[\frac{1+x}{1+x^2}\right] dx = \lim_{n\to\infty} \left[\arctan(x) + \frac{1}{2}\ln(1+x^2)\right]_{-n}^{n}$$

$$\int_{-\infty}^{\infty} \left[\frac{1+x}{1+x^2}\right] dx = 2 \lim_{n\to\infty} \arctan(x)$$

Because $\arctan(x) = -\arctan(x)$,

$$\int_{-\infty}^{\infty} \left[\frac{1+x}{1+x^2}\right] dx = 2 \lim_{n\to\infty} \arctan(x)$$

$$\int_{-\infty}^{\infty} \left[\frac{1+x}{1+x^2}\right] dx = 2\frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \left[\frac{1+x}{1+x^2}\right] dx = \pi$$

However, the function $f(x) = \frac{1+x}{1+x^2}$ is not absolutely integrable on $(-\infty;\infty)$ because we can find a sequence $\{A_n\}_1^\infty$ for which we do not obtain the same limit value of π. Taking for example $A = [-n; 2n]$,

$$\lim_{n\to\infty} \int_{A_n} f = \lim_{n\to\infty} \int_{-n}^{2n} \frac{1+x}{1+x^2}\ dx$$

$$\lim_{n\to\infty} \int_{A_n} f = \lim_{n\to\infty} \left[\arctan(x) + \frac{1}{2}\ln(1+x^2)\right]_{-n}^{2n}$$

$$\lim_{n\to\infty} \int_{A_n} f = \lim_{n\to\infty} \arctan(2n) - \lim_{n\to\infty} \arctan(-n) + \lim_{n\to\infty} \frac{1}{2}\ln\left(\frac{1+4n^2}{1+n^2}\right)$$

$$\lim_{n\to\infty} \int_{A_n} f = \pi + \ln(2)$$

Example 23. [1: p. 278] Consider

$$\int_0^\infty \left[\frac{\sin(x)}{x}\right] dx$$

Since

$$\lim_{x\to 0} \frac{\sin(x)}{x} dx = 1$$

therefore $\frac{\sin x}{x}$ is continuous on $[0;1]$. So now we only need to consider the convergence of

$$\int_1^\infty \frac{\sin(x)}{x}\, dx = \lim_{n\to\infty} \int_1^n \frac{\sin(x)}{x}\, dx$$

by integration by parts,

$$\int_1^\infty \frac{\sin(x)}{x}\, dx = \lim_{n\to\infty} \left\{\left[-\frac{\cos(x)}{x}\right]_1^n - \int_1^n \frac{\cos(x)}{x^2}\, dx\right\}$$

$$\int_1^\infty \frac{\sin(x)}{x}\, dx = \cos(1) - \lim_{n\to\infty} \int_1^n \frac{\cos(x)}{x^2}\, dx$$

Since $\frac{\cos(x)}{x^2}$ is absolutely convergent on $(1;\infty)$, therefore the integral of interest $\int_0^\infty \left[\frac{\sin(x)}{x}\right] dx$ converges, but $f(x) = \frac{\sin(x)}{x}$ is not absolutely integrable on $(0;\infty)$.

Remark 30. [1: p.278] The phenomenon illustrated in the previous few examples, in which $\int_a^b f$ can converge even when f is not absolutely integrable on $(a;b)$, cannot occur when f is nonnegative.

Theorem 61. *[1: p. 278] Suppose that $f : (a;b) \to \mathbb{R}$ is locally integrable with $f \geq 0$. Then f is absolutely integrable iff $\int_a^b$ converges, in which case*

$$\int_{(a;b)} f = \int_a^b f$$

10 Pathlength, line integrals, and differential forms

Remark 31. [1: p. 287] Here we generalize the single-variable integral on a closed in $\mathbb{R}^n$ to get an idea of an integral that is associated with paths in $\mathbb{R}^n$.

Definition 48. [1: p. 287] A $\mathcal{C}^1$ **path** is a continuously differentiable function $\gamma : [a; b] \to \mathbb{R}^n$.

Definition 49. [1: p. 287] A $\mathcal{C}^1$ path is called **smooth** if $\gamma'(t) \neq 0 \; \forall t \in [a; b]$

Example 24. [1: p. 287] The direction of a $\mathcal{C}^1$ path should never change abruptly if its velocity never vanishes. Consider the $\mathcal{C}^1$ path $\gamma = (\gamma_1, \gamma_2) : \mathbb{R} \to \mathbb{R}^2$ defined by

$$x = \gamma_1(t) = t^3$$

and

$$y = \gamma_2(t) = \begin{cases} t^3 & t \geq 0 \\ -t^3 & t \leq 0 \end{cases}$$

The image of γ is the graph $y = |x|$. So there is a corner only at the origin, and that is also the unique point $= 0$ where $\gamma' = 0$.

Remark 32. [1: p. 288] To get a notion of the length of a path, we might approach it with the following approximation in mind: draw a series of cords along the path and sum their lengths, as

$$s(\gamma, \mathcal{P}) = \sum_{i=1}^{k} |\gamma(t_i) - \gamma(t_{i-1})|$$

This motivates the following definition for the actual length of a path.

Definition 50. [1: p. 288] The **length** $s(\gamma)$ of the path $\gamma : [a; b] \to \mathbb{R}^n$ is

$$s(\gamma) = \lim_{|\mathcal{P}| \to 0} s(\gamma, \mathcal{P})$$

whenever this limit exists, so whenever given $\epsilon > 0 \; \exists \delta > 0$ such that $|\mathcal{P}| < \delta \implies |s(\gamma) - s(\gamma, \mathcal{P})| < \epsilon$.

Theorem 62. *[1: p. 288] If $\gamma : [a; b] \to \mathbb{R}^n$ is a $\mathcal{C}^1$ path, then $s(\gamma)$ exists, and*

$$s(\gamma) = \int_a^b |\gamma'(t)| \; dt$$

Example 25. [1: p. 290] Writing $\gamma(t) = (x_1(t), \dots, x_n(t))$ and using Leibniz notation, the path length formula becomes

$$s(\gamma) = \int_a^b \left[\left(\frac{dx_1}{dt} \right)^2 + \dots + \left(\frac{dx_n}{dt} \right)^2 \right]^{\frac{1}{2}} dt$$

Given a $\mathcal{C}^1$ function $f : [a; b] \to \mathbb{R}$, the graph of f is the image of the $\mathcal{C}^1$ path $\gamma : [a; b] \to \mathbb{R}^2$ defined by $\gamma(x) = (x, f(x))$. Substituting $x_1 = t = x$ and $x_2 = y$ into the above formula with $n = 2$, we get the familiar result that the length of the graph of a functrion is

$$s = \int_a^b \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} dx$$

Definition 51. [1: p. 291] The path $\alpha : [a; b] \to \mathbb{R}^n$ is **equivalent** to the path $\beta : [c; d] \to \mathbb{R}^n$ iff there exists a $\mathcal{C}^1$ function

$$\phi : [a; b] \to [c; d]$$

such that, $\forall t \in [a; b]$:

$\hookrightarrow$ $\phi([a; b]) = [c; d]$

$\hookrightarrow$ $\alpha = \beta \circ \phi$

$\hookrightarrow$ $\phi'(t) > 0$

Theorem 63. *[1: p. 291] Suppose that $\alpha : [a; b] \to \mathbb{R}^n$ and $\beta : [c; d] \to \mathbb{R}^n$ are equivalent $\mathcal{C}^1$ paths, and that f is a continuous real-valued function whose domain of definition in $\mathbb{R}^n$ contains the common image of α and β. Then*

$$\int_a^b f(\alpha(t))|\alpha'(t)| \, dt = \int_c^d f(\beta(t))|\beta'(t)|$$

Definition 52. [1: p. 292] A **unit-speed path** $\hat{\gamma}$ is smooth and has $|\hat{\gamma}'| \equiv 1$.

Theorem 64. *[1: p. 292] Every smooth path $\gamma : [a; b] \to \mathbb{R}^n$ is equivalent to a smooth unit-speed path.*

Theorem 65. *[1: p. 295] Given a $\mathcal{C}^1$ path $\gamma : [a; b] \to \mathbb{R}^n$ and n continuous functions $f_1, \dots, f_n$ whose domains of definition in $\mathbb{R}^n$ all contain the image of γ, the* ***line integral*** *$\int_\gamma f_1 \, dx_1 + \cdots f_n \, dx_n$ is defined by*

$$\int_\gamma f_1 \, dx_1 + \dots + f_n \, dx_n = \int_a^b [f_1(\gamma(t))\gamma_1'(t) + \dots + f_n(\gamma(t))\gamma_n'(t)] \, dt$$

10.1 Differential forms

Definition 53. [1: p. 295] A **linear differential form** is a mapping ω on the set $U \in \mathbb{R}^n$ which associates with each point $x \in U$ a linear function $\omega(x) : \mathbb{R}^n \to \mathbb{R}$.

Theorem 66. *[1: p. 295] If ω is a differential form on $U \subset \mathbb{R}^n$, then there exist unique real-valued functions $a_1, \dots, a_n$ on U such that*

$$\omega(x) = \omega_x = a_1(x)\ dx_1 + \cdots a_n(x)\ dx_n$$

for each $x \in U$.

Example 26. [1: p. 296] Let ω be the differential form defined on $\mathbb{R}^2$ minus the origin given by

$$\omega = \frac{-y\ dx + xdx}{x^2 + y^2}$$

If $\gamma_1 : [0;1] \to \mathbb{R}^2 \setminus \{0\}$ is defined by $\gamma_1(t) = \cos(\pi t, \sin \pi t)$, then the image of γ_1 is the upper half of the unit circle, and

$$\int_{\gamma_1} \omega = \int_0^1 \frac{-\sin(\pi t)(-\pi \sin(\pi t)) + \cos(\pi t)(\pi \cos(\pi t))}{\cos^2(\pi t) + \sin^2(\pi t)}\ dt$$

$$\int_{\gamma_1} \omega = \int_0^1 -\sin(\pi t)(-\pi \sin(\pi t)) + \cos(\pi t)(\pi \cos(\pi t))\ dt$$

$$\int_{\gamma_1} \omega = \int_0^1 \pi \sin^2(\pi t) + \pi \cos^2(\pi t)\ dt$$

$$\int_{\gamma_1} \omega = \pi \int_0^1 \sin^2(\pi t) + \cos^2(\pi t)\ dt$$

$$\int_{\gamma_1} \omega = \pi \int_0^1 1\ dt$$

$$\int_{\gamma_1} \omega = \pi$$

However, considering a different path, if $\gamma_2 : [0;1] \to \mathbb{R}^2 \setminus \{0\}$ is defined by $\gamma_2(t) = (\cos(\pi t), -\sin(\pi t))$, then the image of γ_2 is the lower half of the unit circle and is given by

$$\int_{\gamma_1} \omega = \int_0^1 \frac{-(-\sin(\pi t))(-\pi \sin(\pi t)) + \cos(\pi t)(-\pi \cos(\pi t))}{\cos^2(\pi t) + \sin^2(\pi t)}\ dt$$

$$\int_{\gamma_1} \omega = -\int_0^1 \frac{-\sin(\pi t)(-\pi\sin(\pi t)) + \cos(\pi t)(\pi\cos(\pi t))}{\cos^2(\pi t) + \sin^2(\pi t)}\, dt$$

$$\int_{\gamma_1} \omega = -\pi$$

So this is an example in which $\int_{\gamma_1} \omega \neq \int_{\gamma_2} \omega$.

Remark 33. [1: p. 297] Recall that if f is a differentiable real-valued function on the open set $U \in \mathbb{R}^n$, then its differential $df_x(v)$ at $x \in U$ is the linear function on $\mathbb{R}^n$ defined by

$$df_x(v) = D_1 f(x)v_1 + \cdots + D_n f(x)v_n$$

So the differential of a differentiable function is a differential form:

$$df_x = D_1 f(x)dx_1 + \cdots + D_n f(x)dx_n$$

or

$$df = \frac{\partial f}{\partial x_1}dx_1 + \cdots + \frac{\partial f}{\partial x_n}dx_n$$

Definition 54. [3: p. 72] Some alternate terminology from a different book. A differential expression

$$P(x,y)dx + Q(x,y)dy$$

is called an **exact differential** if it is the total differential of a function $f(x,y)$, so if

$$P(x,y) = \frac{\partial}{\partial x} f(x,y)$$

And

$$Q(x,y) = \frac{\partial}{\partial y} f(x,y)$$

Remark 34. We are used to going from functions to their total differentials. The idea of differential forms asks us to think backwards: can we go from a total differential back up to a function?

Theorem 67. *[3: p. 73] A necessary and sufficient condition that the differential form*

$$P(x,y)dx + Q(x,y)dy$$

is exact is that

$$\frac{\partial}{\partial y}P(x,y) = \frac{\partial}{\partial x}Q(x,y)$$

When these functions exist and are continuous.

Theorem 68. *[3: p. 76] We find an appropriate 1-parameter family of solutions by the following equation:*

$$f(x,y) = \int_{x_0}^{x} P(x,y)dx + \int_{y_0}^{y} Q(x_0,y)dy$$

Where x_0 is a point in the region on which the terms are defined.

Example 27. [3: p. 76] Show that the following differential equation (the differential form is the left-hand side) is exact, and find a 1-parameter family of solutions:

$$\cos(y)dy - (x\sin(y) - y^2)dy = 0$$

Assign the following labels:

$$P(x,y) = \cos(y)$$

$$Q(x,y) = -x\sin(y) + y^2$$

Then

$$\frac{\partial}{\partial y}P(x,y) = \frac{\partial}{\partial y}\cos(y)$$

$$\frac{\partial}{\partial y}P(x,y) = -\sin(y)$$

$$\frac{\partial}{\partial x}Q(x,y) = \frac{\partial}{\partial x}\Big(- x\sin(y) + y^2\Big)$$

$$\frac{\partial}{\partial x}Q(x,y) = -\sin(y)$$

Since $\frac{\partial}{\partial y}P(x,y) = \frac{\partial}{\partial x}Q(x,y)$, by Theorem 67, the differential form is exact. To find an appropriate 1-parameter family of solutions, since these are defined on all of $\mathbb{R}^2$, we may take $x_0 = 0$ and $y_0 = 0$. Then $Q(x_0, y)$ is

$$Q(x_0, y) = y^2$$

And by Theorem 68,

$$f(x,y) = \int_0^x \cos(y)dx + \int_0^y y^2 dy$$

$$f(x,y) = x\cos(y) + \frac{y^3}{3}$$

Example 28. [1: p. 297] Let U denote $\mathbb{R}^2$ without the nonnegative x-axis, so $(x,y) \in U$ unless $x \geq 0$ and $y = 0$. Let $\theta : U \to \mathbb{R}$ be the polar angle function

$$\theta(x,y) = \arctan\frac{y}{x}$$

if $x \neq 0$. So

$$D_1\theta(x,y) = \frac{-y}{x^2+y^2}$$

and

$$D_2\theta(x,y) = \frac{x}{x^2+y^2}$$

Then

$$d\theta = \frac{-ydx + xdy}{x^2+y^2}$$

on U. So on its domain of definition, the differential of θ is

$$\omega = \frac{-ydx + xdy}{x^2+y^2}$$

which is defined on $\mathbb{R}^2 \setminus \{0\}$, but because of its definition, the angle function cannot be continuously extended to all of $\mathbb{R}^2 \setminus \{0\}$, so (as the next theorem will state) ω is not the differential of any differentiable function that is defined on all of $\mathbb{R}^2 \setminus \{0\}$.

Theorem 69. *[1: p. 298] If f is a real-valued $\mathcal{C}^1$ function on the open set $U \subset \mathbb{R}^n$, and $\gamma : [a;b] \to U$ is a $\mathcal{C}^1$ path, then*

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a))$$

Corollary 70. *[1: p. 299] If $\omega = df$, where f is a $\mathcal{C}^1$ function on U, and α and β are two $\mathcal{C}^1$ paths in U with the same initial and terminal points, then*

$$\int_\alpha \omega = \int_\beta \omega$$

11 Green's Theorem

Remark 35. [1: p. 304] In this section we treat Green's Theorem, which is a 2 dimensional generalisation of the Fundamental Theorem of Calculus.

11.1 More differential forms

Definition 55. [1: p. 304] Given a $\mathcal{C}^1$ differential form in two variables, $\omega = Pdx + Qdy$, the **differential of the differential form** is

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\ dy$$

We call an expression like this one, of the form $a\ dx\ dy$ with a a function of x and y, a **differential 2-form**.

Definition 56. [1: p. 304] Given a differential 2-form $\alpha = a\ dx\ dy$ with the function a continuous (in which case we simply call α continuous), and a contented set $D \subset \mathbb{R}^2$, the **integral of the differential 2-form** α on D is defined by

$$\int_D \alpha = \int \int_D a(x, y)\ dx\ dy$$

Definition 57. [1: p. 305] We now use **differential 1-form** to refer to differential forms like

$$a_1 dx_1 + \cdots + a_n dx_n$$

or

$$Pdx + Qdy$$

And by the same logic we might clumsily say **differential 0-form** for any real-valued function. Then the differential of a differential 0-form is a differential 1-form, and the differential of a differential 1-form is a 2-form.

Remark 36. [1: p. 305] First let us informally state Green's Theorem.

Let D be a "nice" region in the plane $\mathbb{R}^2$, whose boundary ∂D consists of a finite number of closed curves, each of which is "positively oriented" with respect to D. If $\omega = Pdx + Qdy$ is a $\mathcal{C}^1$ differential 1-form defined on D, then

$$\int_D d\omega = \int_{\partial D} \omega$$

That is,

$$\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\ dy = \int_{\partial D} Pdx + Qdy$$

11.2 Special paths and curves in $\mathbb{R}^2$

Definition 58. [1: p. 305] The continuous path $\gamma : [a;b] \to \mathbb{R}^n$ is called a **piecewise smooth path** if there is a partition $\mathcal{P} = \{a = a_0 < a_1 < \cdots < a_k = b\}$ of the interval $[a;b]$ such that each restriction γ^i of γ to $[a_{i-1}; a_i]$, defined by

$$\gamma^i(t) = \gamma(t) \text{ for } t \in [a_{i-1}; a_i]$$

is smooth.

Definition 59. [1: p. 305] If ω is a continuous differential 1-form, then we define

$$\int_\gamma \omega = \sum_{i=1}^{k} \int_{\gamma^i} \omega$$

Definition 60. [1: p. 306] A **piecewise-smooth curve** C in $\mathbb{R}^n$ is the image of a piecewise-smooth path $\gamma : [a;b] \to \mathbb{R}^n$ which is one-to-one on $(a;b)$.

Definition 61. [1: p. 306] A **closed curve** C in $\mathbb{R}^n$ is a curve on $[a;b]$ such that $\gamma(a) = \gamma(b)$.

Definition 62. [1: p. 306] When a path γ is associated with a curve C, the path is called a **parameterization of the curve**.

Definition 63. [1: p. 306] A pair (C, γ) of a curve C and a path γ is called an **oriented** piecewise smooth curve. In practice often C stands in for (C, γ).

Remark 37. The path, which is parameterized (and therefore captures an idea of "time"), pretty much assigns a sequence to the points on the curve. If that path is piecewise smooth, then you can think of, for example, a particle moving along a curve. This will be how we get an idea of orientation.

Theorem 71. *[1: p. 306] Let $\gamma : [a;b] \to \mathbb{R}^n$ be a piecewise smooth path which is one-to-one and onto on $(a;b)$, and together with a closed curve C it forms an oriented piecewise smooth curve. Then let $\beta : [c;d] \to \mathbb{R}^n$ be a second piecewise smooth path which is one-to-one on $(c;d)$ and with image C. Then we have two cases.*

$$(C, \gamma) = (C, \beta)$$

means that γ and β induce the same orientation of C, provided that their unit tangent vectors are equal at each point where both are defined. In this case, for any continuous differential 1-form ω,

$$\int_\gamma \omega = \int_\beta \omega$$

The other case has

$$(C, \gamma) = -(C, \beta)$$

in which γ and β induce opposite orientations of C, provided that their unit tangent vectors are opposite at each point where both are defined. In this case, for any continuous differential 1-form ω,

$$\int_\gamma \omega = -\int_\beta \omega$$

Definition 64. [1: p. 306] Given an oriented piecewise-smooth curve C and a continuous differential 1-form ω defined on C, we define **the integral of the differential 1-form over the curve** as

$$\int_C \omega = \int_\gamma \omega$$

where γ is any parameterization of C.

Corollary 72. *[1: p. 306] It follows that $\int_C \omega$ is well-defined, and that*

$$\int_{-C} \omega = -\int_C \omega$$

where $C = (C, \gamma)$ and $-C = -(C, \gamma) = (C, \beta)$ whenever γ and β are paths inducing opposite orientations of C.

Definition 65. [1: p. 306] A **nice region** in the plane (recall we are developing Green's Theorem which takes place entirely in $\mathbb{R}^2$) is a connected compact set $D \subset \mathbb{R}^2$ whose boundary ∂D is the union of a finite number of mutually disjoint piecewise-smooth closed curves.

Notation 2. [1: p. 307] A curve of a nice region D has a "positive orientiation" if D remains on your left as you move about the curve according to the specified path. This is equivalent to counterclockwise motion about D. Formalising this takes about a page, which is simply more effort than it's worth for our purposes.

Example 29. [1: p. 308] Given a line integral $\int_C \omega$, where C is an oriented closed curve bounding a nice region D, it is sometimes easier to compute $\int_D \omega$. For example,

$$\int_C 2xydx + x^2dy = \int\int_D \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy)\right]dxdy$$

$$\int_C 2xydx + x^2dy = \int\int_D 0dxdy$$

$$\int_C 2xydx + x^2dy = 0$$

Example 30. [1: p. 308] Sometimes when confronted with an integral like $\int\int_D f(x,y)dxdy$, it can be easier to think of a differential 1-form ω such that $d\omega = f(x,y)dxdy$ and then compute $\int_{\partial D} \omega$. For example if D is a nice region and ∂D is positively oriented, then its area is

$$A = \int\int_D 1dxdy$$

$$A = \int\int_D (-\frac{1}{2}ydx + \frac{1}{2}dy)dxdy$$

$$A = \frac{1}{2}\int_{\partial D} -ydx + xdy$$

and similarly we can obtain

$$A = \int_{\partial D} -ydx$$

$$A = \int_{\partial D} xdy$$

Example 31. [1: p. 308] Suppose that D is the elliptical disk $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$, whose boundary (an ellipse) is parameterized by $x = a\cos(t)$ and $y = b\sin(t)$ for $t \in [0; 2\pi]$. Then its area is

$$A = \frac{1}{2}\int_0^{2\pi} \Big[(-b\sin(t))(-a\sin(t)) + a\cos(t)(b\cos(t))\Big]dt$$

$$A = \frac{1}{2}\int_0^{2\pi} \Big[ab\sin^2(t) + ab\cos^2(t)\Big]dt$$

$$A = \frac{1}{2}\int_0^{2\pi} ab\Big(\sin^2(t) + \cos^2(t)\Big)dt$$

$$A = \frac{1}{2}\int_0^{2\pi} ab dt$$

$$A = \frac{1}{2}[abt]_{t=0}^{2\pi}$$

$$A = \pi ab$$

Example 32. [1: p. 309] Let C be a piecewise smooth curve in $\mathbb{R}^2$ which encloses the origin, and is oriented counterclockwise. Let C_a be a clockwise oriented circle of radius a centered at 0, with a sufficiently small that C_a lies in the interior component of $\mathbb{R}^2 \setminus C$. Then let D be the annular region bounded by C and C_a. If

$$\omega = d\theta = \frac{-ydx + xdy}{x^2 + y^2}$$

Then to get a differential form, apply

$$P = \frac{-y}{x^2 + y^2}dx$$

$$Q = \frac{x}{x^2 + y^2}dy$$

so that

$$\omega = \frac{-y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy$$

so following Definition 55,

$$d\omega = \left(\frac{\partial}{\partial x}\frac{x}{x^2 + y^2} - \frac{\partial}{\partial y}\frac{-y}{x^2 + y^2}\right) dx\ dy$$

$$d\omega = \left((x^2 + y^2)^{-1} - 2x^2(x^2 + y^2)^{-2} + (x^2 + y^2)^{-1} - 2y^2(x^2 + y^2)^{-2}\right) dx\ dy$$

$$d\omega = \left(\frac{x^2+y^2}{(x^2+y^2)^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2}{(x^2+y^2)^2} - \frac{2y^2}{(x^2+y^2)^2}\right) dx\ dy$$
$$d\omega = \left(\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right) dx\ dy$$
$$d\omega = \left(\frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2}\right) dx\ dy$$
$$d\omega = 0$$

Then

$$\int_C d\theta + \int_{C_a} d\theta = \int_{\partial D} \omega = \int_D d\omega = 0$$

Since $\int_{C_a} d\theta = -2\pi$, then

$$\int_C d\theta = -\int_{C_a} = 2\pi$$

It was crucial that C contained the origin, so that all angles were covered. If it did not contain the origin, then

$$\int_C d\theta = 0$$

Definition 66. [1: p. 309] Given a $\mathcal{C}^1$ vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, its **divergence** denoted div $F : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\text{div } F = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}$$

Theorem 73. *[1: p. 310] Let $D \subset \mathbb{R}^2$ be a nice region with ∂D positively oriented, and let N denote the unit outer normal vector to ∂D. From Green's Theorem,*

$$\int_{\partial D} F \cdot N ds = \int_{\partial D} -F_2 dx + F_1 dy$$
$$\int_{\partial D} F \cdot N ds = \int \int_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dx\ dy$$

Then by the definition of divergence,

$$\int_{\partial D} F \cdot N ds = \int \int_D div\ F\ dx\ dy$$

Definition 67. [1: p. 310] The number $\int_{\partial D} F \cdot N ds$ is called the **flux** of the vector field F across ∂D.

Remark 38. See Edwards [1: p. 312-315] for a proof of Green's Theorem. In this reminder document, I just list the vocabulary you need to have in mind when reading that proof.

Definition 68. [1: p. 312] The set $D \subset \mathbb{R}^2$ is called an **oriented (smooth) 2-cell** if there exists a one-to-one $\mathcal{C}^1$ mapping $F : U \to \mathbb{R}^2$, defined on a neighbourhood U of I^2, such that $F(I^2) = D$ and the Jacobian determinant of F is positive at each point of U.

11.3 Pullback

Definition 69. [1: p. 314] Given a $\mathcal{C}^1$ mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$, and using uv coordinates in the doman and xy coordinates in the image so that $F : \mathbb{R}^2_{uv} \to \mathbb{R}^2_{xy}$, and 0-form (so just a real valued function) called $\phi(x, y)$, the **pullback** of the differential 0-form is defined by the composition

$$F^*(\phi) \equiv \phi \circ F$$

So

$$F^*\phi(u, v) = \phi(F_1(u, v), F_2(u, v))$$

Given instead a differential 1-form $\omega = P\ dx + Q\ dy$, we define its **pullback** $F^*\omega = F^*(\omega)$ under F as

$$F^*(\omega) = (F^*P)F^*(dx) + (F^*Q)F^*(dy)$$

where

$$F^*(dx) = \frac{\partial F_1}{\partial u}\ du + \frac{\partial F_1}{\partial v}\ dv$$

and

$$F^*(dy) = \frac{\partial F_2}{\partial u}\ du + \frac{\partial F_2}{\partial v}\ dv$$

This is obtained by carrying out the substitution

$$x = F_1(u, v)$$

$$dx = \frac{\partial x}{\partial u}\ du + \frac{\partial x}{\partial v}\ dv$$

and

$$y = F_2(u, v)$$

$$dy = \frac{\partial y}{\partial u}\ du + \frac{\partial y}{\partial v}\ dv$$

Finally, if instead given the differential 2-form $\alpha = g\ dx\ dy$, the formula is

$$F^*(\alpha) = (g \circ F)(\ \det\ F')\ du\ dv$$

Example 33. [1: p. 314] Let $F : \mathbb{R}^2_{uv} \to \mathbb{R}^2_{xy}$ be given by

$$x = 2u - v$$

and

$$y = 3u + 2v$$

Then

$$|F'| = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}$$

$$|F'| = 7$$

If $\phi(x, y) = x^2 - y^2$, then

$$F^*\phi(u, v) = \phi(x, y) \circ F(u, v)$$

$$F^*\phi(u, v) = (2u - v)^2 - (3u + 2v)^2$$

$$F^*\phi(u, v) = -5u^2 - 16uv - 3v^2$$

Now if $\omega = -y\ dx + 2x\ dy$ then

$$F^*\omega = (F^*P)F^*(dx) + (F^*Q)F^*(dy)$$

Where

$$F^*P = -y$$

$$F^*P = -(3u + 2v)$$

and

$$F^*(dx) = \frac{\partial F_1}{\partial u}\ du + \frac{\partial F_1}{\partial v}\ dv$$

$$F^*(dx) = \frac{\partial(2u - v)}{\partial u}\ du + \frac{\partial(2u - v)}{\partial v}\ dv$$

$$F^*(dx) = 2\ du -\ dv$$

and

$$F^*Q = 2x$$

$$F^*Q = 2(2u - v)$$

and finally

$$F^*(dy) = \frac{\partial F_2}{\partial u}\ du + \frac{\partial F_2}{\partial v}\ dv$$

$$F^*(dy) = \frac{\partial(3u + 2v)}{\partial u}\ du + \frac{\partial(3u + 2v)}{\partial v}\ dv$$

$$F^*(dy) = 3\ du + 2\ dv$$

So that the whole expression is

$$F^*\omega = -(3u + 2v)(2\ du -\ dv) + 2(2u - v)(3\ du + 2\ dv)$$

$$F^*\omega = (6u - 10v)\ du + (11u - 2v)\ dv$$

Now if $\alpha = e^{x+y}\ dx\ dy$, then

$$F^*\alpha = e^{(2u-v)+(3u+2v)}(\ \det\ F'(u, v))\ du\ dv$$

$F^*\alpha = 7e^{5u+v}\ du\ dv$

Theorem 74. *[1: p. 315] Some properties of pullbacks. Let $F : U \to \mathbb{R}^2$ be as it is in the definition of an oriented 2-cell $D = F(I^2)$, and let ω be a $\mathcal{C}^1$ differential 1-form and α be a $\mathcal{C}^1$ differential 2-form on D. Then pullback have the following three properties:*

- ↪ $\int_{\partial D} \omega = \int_{\partial I^2} F^*\omega$
- ↪ $\int_D \alpha = \int_{I^2} F^*\alpha$
- ↪ $d(F^*\omega) = F^*(d\omega)$

11.4 Green's theorem for cellulated nice regions

Theorem 75. *[1: p. 315] Green's Theorem for oriented 2-cells: If D is an oriented 2-cell and ω is a $\mathcal{C}^1$ differential 1-form on D, then*

$$\int_D d\omega = \int_{\partial D} \omega$$

Remark 39. [1: p. 316] A more general nice region might be decomposable into oriented 2-cells so that the truth of Green's theorem for each of these oriented 2-cells implies its truth for all of D.

Example 34. [1: p. 316] Consider an annular region D that is bounded by 2 concentric circles. D is the union of two oriented 2-cells D_1 and D_2 (see the text). We can carve it up into paths $\gamma_1, \cdots, \gamma_6$ as done in the text, such that $\partial D_1 = \gamma_2 - \gamma_6 + \gamma_3 - \gamma_5$ and $\partial D_2 = \gamma_1 + \gamma_5 + \gamma_4 + \gamma_6$, where the signs represent opposite orientations along each path. Applying Green's theorem then to each of D_1 and D_2,

$$\int_{D_1} d\omega = \int_{\gamma_2} \omega - \int_{\gamma_6} \omega + \int_{\gamma_3} \omega - \int_{\gamma_5} \omega$$

and

$$\int_{D_2} d\omega = \int_{\gamma_1} \omega + \int_{\gamma_5} \omega + \int_{\gamma_4} \omega + \int_{\gamma_6} \omega$$

Now since these are disjoint paths, adding these gives

$$\int_D d\omega = \int_{D_1} d\omega + \int_{D_2} d\omega$$

$$\int_D d\omega = \int_{\gamma_2} \omega - \int_{\gamma_6} \omega + \int_{\gamma_3} \omega - \int_{\gamma_5} \omega + \int_{\gamma_1} \omega + \int_{\gamma_5} \omega + \int_{\gamma_4} \omega + \int_{\gamma_6} \omega$$

$$\int_D d\omega = \int_{\gamma_2} \omega + \int_{\gamma_3} \omega + \int_{\gamma_1} \omega + \int_{\gamma_4} \omega$$

$$\int_D d\omega = \left(\int_{\gamma_1} \omega + \int_{\gamma_2} \omega \right) + \left(\int_{\gamma_3} \omega + \int_{\gamma_4} \omega \right)$$

Which, consulting the labels on the diagram in the text, is exactly

$$\int_D d\omega = \int_{\partial D} \omega$$

Validating Green's theorem for the annular region D.

Definition 70. [1: p. 317] Example 34 is one instance of something more general: a smooth **cellulation** of a nice region D is a finite collection $\mathcal{K} = \{D_1, \dots, D_k\}$ of oriented 2-cells such that

$$D = \bigcup_{i=1}^{k} D_i$$

A **cellulated nice region** is a nice region D together with a cellulation $\mathcal{K}$ of D.

Theorem 76. *[1: p. 317] Green's Theorem: If $D \subset \mathbb{R}^2$ is a cellulated nice region and ω is a $\mathcal{C}^1$ differential 1-form on D, then*

$$\int_D d\omega = \int_{\partial D} \omega$$

Theorem 77. *[1: p. 318] If $\omega = P\ dx + Q\ dy$ is a $\mathcal{C}^1$ differential 1-form defined on $\mathbb{R}^2$, then the following three conditions, which harken back to Definition 54 of exact differential forms and its associated Theorems 67 and 68, are equivalent:*

- ↪ *There exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $df = \omega$*
- ↪ $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ *on $\mathbb{R}^2$*
- ↪ *Given points a and b, the integral $\int_\gamma \omega$ is independent of the piecewise smooth path γ from a to b.*

Example 35. [1: p. 319] If $\omega = y\ dx + x\ dy$, then the function f defined as

$$f(x_0, y_0) = \int_{\gamma(x_0,y_0)} y\ dx + x\ dy$$

$$f(x_0, y_0) = \int_0^1 (y_0 t)(x_0\ dt) + (x_0 t)(y_0\ dt)$$

$$f(x_0, y_0) = x_0 y_0 \int_0^1 2t\ dt$$

$$f(x_0, y_0) = x_0 y_0$$

So $f(x, y) = xy$.

References

[1] C. H. Edwards. *Advanced calculus of several variables.* Dover, 1973.

[2] Ruowen Liu. *Real analysis course notes.* 2020.

[3] Morris Tenenbaum and Harry Pollard. *Ordinary differential equations.* Dover, 1963.