



Political Science Math Camp

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Contributors

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1 Precursors and notation

1.1 The beginning: Names for numbers

Some numbers are for counting; we call these the natural numbers, \mathbb{N} . These are numbers like $1, 2, 3, \dots$

The natural numbers, their negatives, and zero are called the integers, \mathbb{Z} . These are numbers like $\dots, -3, -2, -1, 0, 1, 2, 3, \dots$

Alongside the integers are numbers obtained by dividing an integer by an integer. We call these the rational numbers, \mathbb{Q} . These are represented by fractions and decimals, and include numbers like $-\frac{30}{100}, 0, 1$, and 5.8 . The only integer we cannot divide a number by is 0 .

Together with numbers that cannot be represented as a fraction or a finite decimal (irrational numbers like π), all of these numbers form the real numbers, \mathbb{R} .

We will care a lot about series and sequences of numbers, which are arbitrarily long lists of numbers.

1.2 Fractions

Fractions, which are a way of representing rational numbers (a *ratio* of numbers), have two parts, a numerator and a denominator, arranged like this:

$$\frac{\text{numerator}}{\text{denominator}}$$

Let's talk about how to add, subtract, multiply, and divide fractions. The simplest is actually multiplication; when we multiply two fractions, we just multiply their numerators and their denominators. So,

$$\frac{7}{8} \cdot \frac{3}{4} = \frac{7 \cdot 3}{8 \cdot 4}$$

$$\frac{7}{8} \cdot \frac{3}{4} = \frac{21}{32}$$

If a series of fractions have the same denominator, we can simply add or subtract their numerators, like this:

$$\frac{3}{18} + \frac{7}{18} = \frac{10}{18}$$

It's much easier to understand fractions when the numerators and denominators are as small as they can be. To do this, we simplify by dividing out any common terms like this:

$$\frac{10}{18} = \frac{5 \cdot 2}{9 \cdot 2}$$

$$\frac{10}{18} = \frac{5}{9} \cdot \frac{2}{2}$$

$$\frac{10}{18} = \frac{5}{9} \cdot 1$$

$$\frac{10}{18} = \frac{5}{9}$$

To get two fractions' denominators to be equal so that we can add or subtract them, one idea is to multiply each fraction by the other fraction's denominator divided by itself (which simplifies to 1). This is rarely the simplest way of doing things, but it always works. For example:

$$\frac{7}{18} - \frac{2}{15} = \frac{7}{18} \cdot \frac{15}{15} - \frac{2}{15} \cdot \frac{18}{18}$$

$$\frac{7}{18} - \frac{2}{15} = \frac{105}{270} - \frac{36}{270}$$

$$\frac{7}{18} - \frac{2}{15} = \frac{69}{270}$$

$$\frac{7}{18} - \frac{2}{15} = \frac{23}{90}$$

Finally, when dividing fractions, we multiply by the reciprocal.

$$\frac{\frac{7}{18}}{\frac{5}{6}} = \frac{7}{18} \cdot \frac{6}{5}$$

$$\frac{\frac{7}{18}}{\frac{5}{6}} = \frac{42}{90}$$

$$\frac{\frac{7}{18}}{\frac{5}{6}} = \frac{7}{15}$$

Practice problems: Give all of the answers in the simplest form

$$\frac{7}{17} - \frac{4}{15} = \frac{37}{255}$$

$$\frac{7}{10} \cdot \frac{2}{15} = \frac{7}{75}$$

$$\frac{12}{8} \div \frac{5}{3} = \frac{9}{10}$$

Simplify $\frac{8\pi}{2\pi} = 4$

1.3 Inequalities

- $x > y$: “ x is greater than y ”
- $x < y$: “ x is less than y ”
- $x \geq y$: “ x is greater than or equal to y ”
- $x \leq y$: “ x is less than or equal to y ”

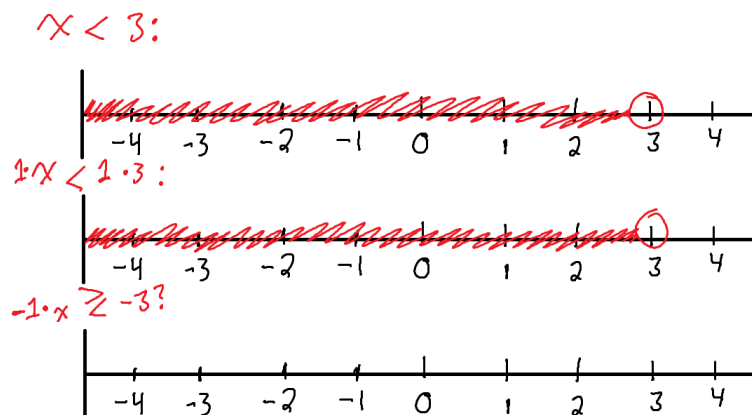
Addition, subtraction, and multiplication for inequalities works just like it does for equalities. But if you multiply or divide both sides of an inequality by a negative value, you have to flip the sign of the inequality. For example:

$$3x < 4x + 2$$

$$-x < 2$$

$$x > -2$$

Why?



1.4 Absolute Value

We can think of numbers as having two components: sign and magnitude. The sign of a number is either positive (+) or negative (-) – or, in the case of 0, neither. The magnitude of a number is how distant it is from 0. For example, -2 and 2 have opposite signs but equal magnitudes, because they are equally far from 0.

The absolute value expression, written $|x|$, takes any number x and returns the magnitude of the number with a positive sign. So, $|x| = |-x|$. For example, $|-2| = 2$.

When “solving” for an expression inside absolute value signs, we are asking: what number could have been put into the absolute value expression to return a certain value? There will always be a positive and a negative answer (except in the case of 0). So when solving an equation for the absolute value of a variable, we have to solve for the case where that variable is positive and the case where that variable is negative.

$$|x - 3| > 4 \Rightarrow x - 3 > 4 \text{ and } -(x - 3) > 4$$

$$x > 7 \text{ or } x < -1$$

Notation: \Rightarrow means “implies”.

1.5 Summation: Σ

The sum of a sequence can be written as a summation (Σ). This saves us from having to write down arbitrarily large sequences.

For example, the sum of all the natural numbers from 1 to 100 can be written as:

$$\sum_{i=1}^{100} i = 1 + 2 + \cdots + 100$$

The bottom of the Σ symbol indicates an index (here, i), and its start value 1. At the top is where the index ends. The content to the right of the summation is the terms that are being added.

Summation notation is a concise way to represent large (or infinite) sums:

$$\sum_{k=0}^6 2x^k = 2x^6 + 2x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 2$$

Just like in the longhand form, you can factor out constants:

$$\sum_{k=0}^6 cx_k = c \sum_{k=0}^6 x_k$$

or reorder terms:

$$\sum_{i=1}^n x_i + \sum_{i=1}^n y_i = \sum_{i=1}^n (x_i + y_i)$$

though it's ugly, you'll also sometimes see:

$$\sum_{i=1}^n c = nc$$

1.6 Product: Π

The product of a sequence of term can be written in product notation (Π).

$$\prod_{i=1}^n x_i = x_1 \cdot x_2 \cdot x_3 \cdot \dots \cdot x_n$$

The product of the integers between 1 and 4 is:

$$\prod_{i=1}^4 i = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$

Properties:

$$\prod_{i=1}^n cx_i = c^n \prod_{i=1}^n x_i$$

For $k < n$,

$$\prod_{i=k}^n cx_i = c^{n-k+1} \prod_{i=k}^n x_i$$

$$\prod_{i=1}^n c = c^n$$

No idea why someone would write that last one rather than just an exponent, but people do all sorts of things that I don't understand.

1.7 Example questions

In the following questions, where relevant, let $x_1 = 4$, $x_2 = 3$, $x_3 = 7$, $x_4 = 11$, and $x_5 = 2$.

1. $\sum_{i=1}^5 i = 1 + 2 + 3 + 4 + 5 = 15$

2. $\prod_{i=1}^5 i = 1 * 2 * 3 * 4 * 5 = 120$

3. $\sum_{i=1}^3 7x_i = 7(4 + 3 + 7) = 98$

4. $\sum_{i=1}^5 2 = 2 + 2 + 2 + 2 + 2 = 10$

5. $\prod_{i=3}^5 (2)x_i = 2^3(7)(11)(2) = 1232$

2 Functions

2.1 Definition of a function

A **function** f takes each element in the set X (called its domain) and assigns it a unique element in the set Y (called its codomain). We write that as $f : X \rightarrow Y$, and say “the function f maps each element of X to an element of Y ”.

Example: Say that X is the set of natural numbers, \mathbb{N} , and Y is the set of real numbers, \mathbb{R} . Then an example of a function is $f(n) = \frac{n}{2}$. **Question:** Reassure yourself that every value this function outputs will be a real number. Then check that it’s a function.

Example that doesn’t work: We can never allow a function to have $f(x) = y_1$ and $f(x) = y_2$ for $y_1 \neq y_2$. A function can’t take one number x and map it onto two different numbers, y_1 and y_2 . So a legal function is

$$f(x) = \begin{cases} x & x \neq 2 \\ 3 & x = 2 \end{cases}$$

But an illegal function would be

$$f(x) = \begin{cases} x & x \neq 2 \\ 3 & x = 2 \\ 4 & x = 2 \end{cases}$$

Notation: Functions represented this way are called “piecewise functions”.

Rule: You might remember the **vertical line test**. If we draw out an example of a mapping that violates the definition of a function, with the X values on the x -axis and the Y values on the y -axis, then we’ll be able to put a vertical line somewhere such that it touches two values of y . If we can do that, we’re not looking at a function.

When f takes each value in X onto exactly one value in Y , we call f “one-to-one”. We can also imagine “many-to-one” functions, like $f(x) = 2$ for $x \in \mathbb{R}$. **Question:** Graph this function and verify that it passes the vertical line test. **Question:** If $f(x) = 2$ is a “many-to-one” function, try to define a “one-to-many” function. Can you do it?

2.2 Multivariable functions

We will often care about functions that map from combinations of numbers onto combinations of numbers, which are called multivariable functions.

Notation: we can write \mathbb{R}^1 instead of \mathbb{R} , simply denoting the set of real numbers. Then \mathbb{R}^2 denotes the set of *pairs* of real numbers, \mathbb{R}^3 is the set of triplets of real numbers, and so on. In general, \mathbb{R}^n is called an n -tuple of real numbers. We often talk about these as being coordinates in an n -dimensional space of real numbers, so an n -tuple is a point in n -space.

Examples:

- $5 \in \mathbb{R}^1$
- $(-33.4, 13) \in \mathbb{R}^2$
- $(42, \pi, \frac{11}{3}) \in \mathbb{R}^3$

Some notation: $x \in X$ means that x is an **element** of the set X .

To write a function that maps one variable onto one variable, we can write $f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$

Example: $f(x) = x + 2$. For each x in \mathbb{R}^1 , $f(x)$ assigns the number $x + 2$.

You'll very commonly see functions that map two variables onto two variables, like $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

Example ($f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$): $f(x, y) = x^2 + y^2$. For each ordered pair (x, y) in \mathbb{R}^2 , $f(x, y)$ assigns the number $x^2 + y^2$. Notice that $x^2 + y^2$ is always just a number in \mathbb{R}^1 , not a pair.

Example ($f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$): $f(x, y) = (x^2, y^2)$

So we can have zany functions that map n -tuples onto m -tuples, for $n, m \in \mathbb{N}$. So for example $f : \mathbb{R}^2 \rightarrow \mathbb{R}^4$, like $f(x, y) = (x + y, x - y, x^2 + \pi, -\frac{\sqrt{x}}{\sqrt{y}})$. This gets messy in a hurry.

2.3 Domain, Range, and Image

Some functions are defined only on proper subsets of (sets contained within) \mathbb{R}^n .

Domain: the set of numbers x in X at which $f(x)$ is defined.

Range: elements of Y assigned by $f(x)$ from elements of X , or

$$f(X) = \{y : y = f(x), x \in X\}$$

This is most often used when talking about a function $f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$.

Image: same as range, but more often used when talking about a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$.

2.4 Injection, Surjection, and Bijection

A correspondence c from set S to T is called **1 to 1** or **injective** if c never maps multiple elements of S onto the same element of T .

Say $S = \{1, 2, 3\}$, $T = \{A, B, C, D\}$, and we have

$$c(1) = A$$

$$c(2) = B$$

$$c(3) = C$$

This is a 1-1 function.

Say $S = \{1, 2, 3\}$, $T = \{A, B, C, D\}$, and we have

$$c(1) = A$$

$$c(2) = B$$

$$c(3) = B$$

This is not a 1-1 function.

A correspondence c from set S to set T is called **onto** or **surjective** if every element in T is mapped to by at least one element in S .

Say $S = \{1, 2, 3, 4\}$, $T = \{A, B, C\}$, and we have

$$c(1) = A$$

$$c(2) = B$$

$$c(3) = C$$

$$c(4) = C$$

This is an onto function.

Is it 1-1? **Answer:** No, because 3 and 4 both output C .

Say $S = \{1, 2, 3\}$, $T = \{A, B, C, D\}$, and we have

$$c(1) = A$$

$$c(2) = B$$

$$c(3) = C$$

This is not an onto function.

Is it 1-1? **Answer:** Yes.

A set relation is called **bijective** (or, simply, **1 to 1 and onto**) if it is injective (1 to 1) and surjective (onto).

Were any of the examples so far 1-1 and onto? **Answer:** No.

Say $S = \{1, 2, 3, 4\}$, $T = \{A, B, C, D\}$, and we have

$$\begin{aligned}c(1) &= A \\c(2) &= B \\c(3) &= C \\c(4) &= D\end{aligned}$$

This is a 1 to 1 and onto function.

Question: Pick any two sets of different sizes, say $S = \{1, 2, 3\}$, $T = \{A, B, C, D\}$. Find a bijective mapping from S to T . **Answer:** It's not possible. If you've found a bijective mapping between two sets, it means they have the same number of elements.

Question: Connecting this back to functions, consider $f : \mathbb{N} \rightarrow \mathbb{N}$ defined by $f(n) = n^2$. Is it bijective? **Answer:** It is 1 to 1, because every natural number has a square, but it is not onto; for example, no natural number squared equals 5; $\sqrt{5} \notin \mathbb{N}$.

The **inverse** of a correspondence c , written c^{-1} , is the correspondence that undoes c . We'll talk a lot about inverse functions in the methods sequence.

2.4.1 Example questions

For each of the following, state whether they are one-to-one or many-to-one functions.

1. For $x \in [0, \infty]$, $f(x) = x^2$ (one-to-one)
2. For $x \in [-\infty, \infty]$, $f(x) = x^2$ (many-to-one)
3. For $x \in [-3, \infty]$, $f(x) = x^2$ (many-to-one)
4. For $x \in [0, \infty]$, $f(x) = \sqrt{x}$ (one-to-one)

2.5 Exponential Functions

Exponential functions have the form

$$f(x) = a^x$$

where a is the **base**, usually $a \in \mathbb{R}$, and x is a variable. Such functions represent situations where growth is proportional to size.

Example: Suppose a country's GDP is currently 1 dollar, and it grows at 2% every year. After 1 year, GDP will be

$$(1 + 0.02) = 1.02^1$$

After 2 years, GDP will be

$$(1 + 0.02)(1 + 0.02) = 1.02^2$$

After x years, GDP will be

$$1.02^x$$

Properties:

- $a^x a^y = a^{x+y}$

When multiplying the same base but different exponents, add the exponents.

$$x^3 \cdot x^4 = x^7$$

- $a^x / a^y = a^{x-y}$

When dividing the same base but different exponents, subtract the exponents.

$$\frac{x^5}{x^2} = x^3$$

- $(a^x)^y = a^{xy}$

When taking an exponent to the power of some other value, multiply the exponents.

$$(x^3)^3 = x^9$$

But notice this is different from x^{3^3}

- $a^{-x} = \frac{1}{a^x}$

Negative exponents can be expressed in the denominators of fractions.

$$x^{-2} = \frac{1}{x^2}$$
$$x^{-3} \cdot x^4 = \frac{x^4}{x^3}$$

- $\sqrt[a]{x} = x^{\frac{1}{a}}$

Roots can be expressed as fractional exponents.

$$\sqrt{x} = x^{\frac{1}{2}}$$

- $a^0 = 1 \quad \forall a$

Notation: \forall means “for all”, \exists means “there exists”.

2.5.1 Example questions

1.

$$\begin{aligned} & \frac{(6x^3y^{-4})^{-2}}{(3x^2y^5)^{-3}} \\ &= \frac{6^{-2}x^{-6}y^8}{3^{-3}x^{-6}y^{-15}} \\ &= \frac{y^83^3x^6y^{15}}{6^2x^6} \\ &= \frac{3^3x^6y^{23}}{6^2x^6} \\ &= \frac{3^3y^{23}}{6^2} \\ &= \frac{27y^{23}}{36} \\ &= \frac{3y^{23}}{4} \end{aligned}$$

2.

$$\begin{aligned} & \frac{(-3x^{-4}y)^{-4}}{(5x^{-2}y^3)^0} \\ &= \frac{(-3x^{-4}y)^{-4}}{1} \\ &= \frac{(-3)^{-4}x^{16}y^{-4}}{1} \\ &= \frac{x^{16}}{(-3)^4y^4} \end{aligned}$$

$$= \frac{x^{16}}{81y^4}$$

3.

$$\left(\frac{5x^{-4}y^2}{3x^{-1}y^{-3}}\right)^{-2}$$

$$= \frac{5^{-2}x^8y^{-4}}{3^{-2}x^2y^6}$$

$$= \frac{x^83^2}{5^2y^4x^2y^6}$$

$$= \frac{3^2x^8}{5^2x^2y^{10}}$$

$$= \frac{3^2x^6}{5^2y^{10}}$$

$$= \frac{9x^6}{25y^{10}}$$

4.

$$\left(\frac{7x^3y}{2x^{-5}y^2}\right)^0 \cdot (4x^{-3}y^2)^{-1}$$

$$= (4x^{-3}y^2)^{-1}$$

$$= 4^{-1}x^3y^{-2}$$

$$= \frac{x^3}{4^1y^2}$$

$$= \frac{x^3}{4y^2}$$

5.

$$\begin{aligned} & \frac{(8x^3y^{-4})^{-2}}{(-4x^{-1}y)^{-3}(2x^5y^{-3})^{-2}} \\ &= \frac{8^{-2}x^{-6}y^8}{(-4)^{-3}x^3y^{-3}2^{-2}x^{-10}y^6} \\ &= \frac{y^8(-4)^3y^32^2x^{10}}{8^2x^6x^3y^6} \\ &= \frac{(-4)^32^2x^{10}y^{11}}{8^2x^9y^6} \\ &= \frac{(-4)^32^2xy^3}{8^2} \\ &= \frac{-64 \cdot 4xy^5}{64} \\ &= \frac{-256xy^5}{64} \\ &= -4xy^5 \end{aligned}$$

2.6 Polynomials

A **monomial** is any function of the form $f(x) = ax^k$, where k is a nonnegative integer, a is the coefficient, k is the degree. For example, $y = x^2$, $y = -\frac{1}{2}x^3$

A **polynomial** is a sum of (finitely many) monomials. For example, $y = -\frac{1}{2}x^3 + 3x^2 + x + 2$, or $y = 3x + 5$.

The **degree** of a polynomial is equal to its largest exponent (highest degree of its monomial terms). The degree of the example above is 3. Conventionally we write polynomials with terms in decreasing degree.

A very important problem is solving for the value(s) of x satisfying $f(x) = 0$. (This is called “finding roots”, and we’ll do it often when we get to maximization.) There are two methods to do this for high-degree polynomials:

1. If f is a polynomial of degree 2, use the Quadratic Equation!

- $ax^2 + bx + c = 0$ implies $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

2. Factor the polynomial.

$$x^3 + 2x^2 + x = 0$$

$$x^3 + 2x^2 + x = x(x^2 + 2x + 1)$$

$$x^3 + 2x^2 + x = x(x + 1)(x + 1)$$

$$\text{so } x^3 + 2x^2 + x \text{ iff } x = 0 \text{ or } x + 1 = 0 \implies x = -1$$

2.7 Logarithms

The log function can be thought of as an inverse for exponential functions. a is referred to as the “base” of the logarithm. The **base a logarithm** of y is the exponent to which we must raise a in order to get y . There are two common logarithms: base 10 and base e . For example, the statement $y = \log_{10}(x)$ means that y is the value that we should raise 10 to in order to obtain x . So,

1. Base 10: $y = \log_{10}(x) \iff 10^y = x$. The base 10 logarithm is often simply written as “log(x)” with no base denoted.
2. Base e : $y = \log_e(x) \iff e^y = x$. The base e logarithm is referred to as the “natural” logarithm and is written as $\ln(x)$, pronounced “lawn”.

2.7.1 Properties of logarithmic functions with any base

- $\log_a(a^x) = x$
- $a^{\log_a(x)} = x$
- $\log(xy) = \log(x) + \log(y)$
- $\log(x^y) = y \log(x)$
- $\log\left(\frac{1}{x}\right) = \log(x^{-1}) = -\log(x)$
- $\log\left(\frac{x}{y}\right) = \log(x \cdot y^{-1}) = \log(x) + \log(y^{-1}) = \log(x) - \log(y)$
- $\log(1) = \log(a^0) = 0$

2.7.2 Change of base

You can use the change of base formula to switch bases as necessary:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

Example:

$$\log_{10}(x) = \frac{\ln(x)}{\ln(10)}$$

You can use logs to switch between sum and product notation, which will be important when learning Maximum Likelihood Estimation in POLSCI 599. The log of a product is equal to the sum of the logs.

$$\begin{aligned}\log\left(\prod_{i=1}^n x_i\right) &= \log(x_1 \cdot x_2 \cdot x_3 \dots x_n) \\ &= \log(x_1) + \log(x_2) + \log(x_3) + \dots + \log(x_n) \\ &= \sum_{i=1}^n \log(x_i)\end{aligned}$$

2.7.3 Example questions

Solve the following three logarithms

1. $\log_4(16) = \log_4(4^2) = 2$
2. $\log_2(16) = \log_2(2^4) = 4$
3. $\log_{\frac{3}{2}}\left(\frac{27}{8}\right) = \log_{\frac{3}{2}}\left(\frac{3^3}{2^3}\right) = 3$

Apply the logarithm rules we've discussed to write the following expressions as sums of logarithms.

1. $\log_4(x^3 y^5) = \log_4(x^3) + \log_4(y^5) = 3 \log_4(x) + 5 \log_4(y)$
2. $\log\left(\frac{x^9 y^5}{z^3}\right) = \log(x^9) + \log(y^5) - \log(z^3) = 9 \log(x) + 5 \log(y) - 3 \log(z)$
3. $\ln \sqrt{xy} = \ln \sqrt{x} + \ln \sqrt{y} = \ln x^{\frac{1}{2}} + \ln y^{\frac{1}{2}} = \frac{1}{2}(\ln x + \ln y)$

2.8 Composite Functions

$$f : X \rightarrow Y$$

$$g : Y \rightarrow Z$$

$$f \circ g : X \rightarrow Z$$

The composition $f \circ g$ is the same as writing $f(g(y))$. For example,

$$g(x) = x^2 + 5x - 3$$

$$h(y) = 3(y - 1)^2 - 5$$

$$(h \circ g)(-6) = ?$$

$$g(-6) = (-6)^2 + 5(-6) - 3 = 3$$

$$h(3) = 3(3 - 1)^2 - 5 = 7$$

This is the same as $h(g(-6))$.

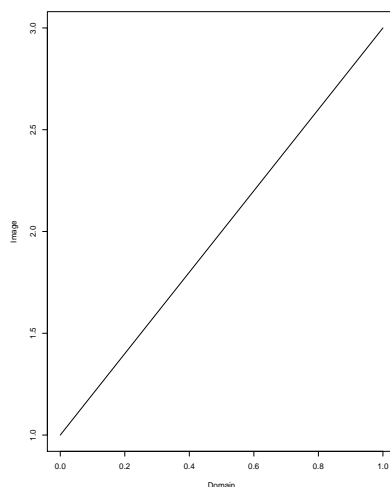
2.9 Functions Over Real Numbers (Plots!)

Let's take a moment to mention the plots of functions that you're familiar with. Take an example of

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

Example:

$$f(x) = 2x + 1$$



2.10 Inverse Functions

Similar to the idea we mentioned when talking about correspondences, the inverse of a function is the mapping that “undoes” that function: where $f : X \rightarrow Y$ takes a value x and returns value y , its inverse $f^{-1} : Y \rightarrow X$ takes the same value y and returns the original value x . So the inverse function always satisfies the property:

$$f^{-1}(f(x)) = f^{-1}(y) = x$$

The usual procedure for obtaining an inverse of a single variable function is to just switch y and x in the equation and then isolate for x . So returning to the function above, $y = 2x + 1$, we can obtain the inverse by doing the following:

$$x = 2y + 1$$

$$x - 1 = 2y$$

$$y = \frac{x - 1}{2}$$

Now let's use our labels and see if the procedure did what we said it should:

$$f(x) = 2x + 1$$

$$f^{-1}(x) = \frac{x - 1}{2}$$

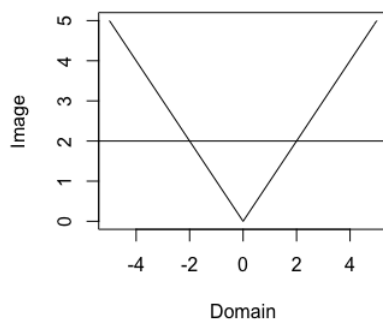
We said that it should always be true that $f^{-1}(f(x)) = x$, so let's try it:

$$f^{-1}(f(x)) = \frac{(2x + 1) - 1}{2}$$

$$f^{-1}(f(x)) = \frac{2x}{2}$$

$$f^{-1}(f(x)) = x$$

In this case it worked, but unfortunately this isn't a one-size-fits-all-functions procedure, because not all functions have inverse functions!



Question: What's the visual equivalent of the trick we pulled, switching the x and the y ? Try to switch the position of the x and y values in the plot and see if you can tell why this function has no inverse function.

Question: Draw another valid function that doesn't have an inverse function.

2.11 Increasing/Decreasing Functions

Increasing:

$$b > a \implies f(b) \geq f(a)$$

Strictly Increasing:

$$b > a \implies f(b) > f(a)$$

Decreasing:

$$b > a \implies f(b) \leq f(a)$$

Strictly Decreasing:

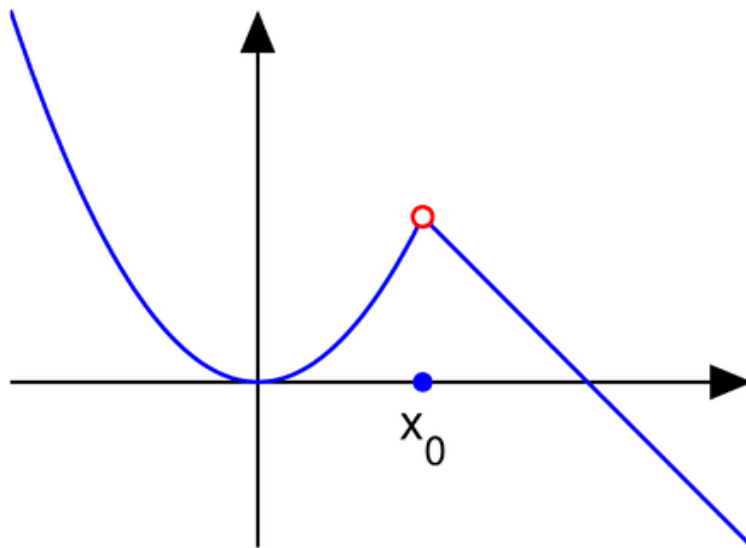
$$b > a \implies f(b) < f(a)$$

A **monotonic** function is either increasing or decreasing everywhere. A strictly monotonic function is either strictly increasing or strictly decreasing everywhere. All strictly monotonic functions have inverses.

Question: Try to draw a strictly monotonic function that doesn't have an inverse. Why can't you do it?

2.12 Continuous Functions

We'll talk a lot more about this later, but for now let's roughly state: a function is **continuous** if there are no breaks (aka "discontinuities"). Can't look like **this**:



The graph above is an example of a **piecewise function**.

$$f(x) = \begin{cases} x^2 & x < 1 \\ 0 & x = 1 \\ 2 - x & x > 1 \end{cases}$$

3 Sets

Definition: A **set** is a collection of distinguishable objects. The objects in a set are called **members** or **elements**. We think of the set itself as one coherent object.

$$A = \{3, \textit{apple}, Q, 42\}$$

$$B = \{3, \textit{elephant}, Q, \textit{purple}\} = \{\textit{elephant}, Q, 3, \textit{purple}\}$$

$$A \neq B$$

$$\mathbb{N}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}, \dots$$

3.1 Describing a Set

- Cardinality: the number of elements in a set. In the example above, $|A| = |B| = 4$
- Finiteness: finite sets have a specified number of elements; infinite sets have no limit on the number of elements
- Countability: countable sets have elements that can be enumerated (e.g. the integers); uncountable sets' elements cannot be enumerated (i.e. real numbers from 0 to 1)
- Order: Sets themselves do not capture any notion of orderedness; $\{0, 1\} = \{1, 0\}$. But there are clever ways to use multiple sets to enforce an order among the elements. So for the rest of the notes we're going to finesse this a bit and talk about "ordered sets", which are sets that we've imposed an order on. We often use these to model situations where order matters, like in a preference ranking

3.2 Special Sets

Natural Numbers:

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

Note: sometimes 0 is included in the natural numbers.

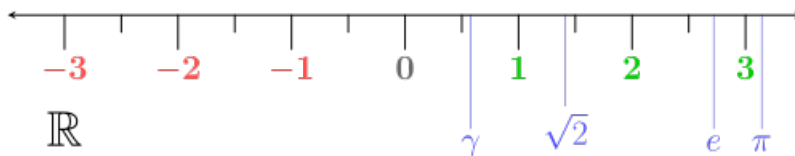
Integers:

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

Real Numbers (\mathbb{R})

The Empty Set:

$$\emptyset = \{\}$$



3.3 Set Notation

English: “ x is a member of the set S .”

Math: $x \in S$

English: “ x is *not* a member of the set S .”

Math: $x \notin S$

English: “There exists some element x ”

Math: $\exists x$

English: “For all x ”

Math: $\forall x$

English: “The set of all integers greater than 4.”

Math: $\{x \in \mathbb{Z} : x > 4\}$

Or: $\{x \in \mathbb{Z} | x > 4\}$

(I’d get out of the latter habit soon, though, because starting in POLSCI 599 you’ll very often use vertical bars inside expressions to denote “or”).

English: “All of the elements in the set S are less than 3.”

Math: $\forall x \in S, x < 3$

Question: Translate the following sentences into set notation:

- “The set of all numbers that are less than eight and at least nine”. How many elements are in this set? **Answer:** $\{n \in \mathbb{N} : n < 8, n > 9\} = \emptyset$ and $|\emptyset| = 0$
- “The set of all functions that have a function inverse” **Answer:** $\{f(x) : \exists f^{-1}(x)\}$
- “The set of all x such that $x = 3$ ”. **Answer:** $\{x : x = 3\} = \{3\} \neq 3$
- “The set of all real numbers x and y such that $x + y = 2$ ” **Answer:** $\{x, y \in \mathbb{R} : x + y = 2\}$

3.4 Subsets

Definition: For sets A and B , A is a **subset** of B if every element of A is also an element of B .

$$\{3, Q\} \subseteq \{3, Q\}$$

Definition: For sets A and B , A is a **proper subset** of B if A is a subset of B and $A \neq B$. So, $\exists b \in B$ such that $b \in B$ and $b \notin A$.

$$\{3\} \subset \{3, Q\}$$

3.5 Set operations

Intersection:

$$A \cap B = \{3, Q\}$$

Union:

$$A \cup B = \{3, apple, elephant, Q, 42, purple\}$$

Difference:

$$A \setminus B = \{apple, 42\}$$

$$B \setminus A = \{elephant, purple\}$$

Symmetric difference:

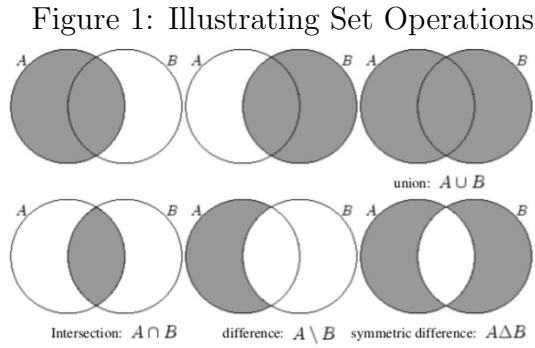
$$A \setminus B \cup B \setminus A = \{apple, 42, elephant, purple\}$$

Complement: A^c is everything that isn't in A (usually out of some much larger set; in POLSCI 599 you'll learn about the set of events X as a subset of all of the events Ω that could possibly have happened in, eg, a statistical experiment)

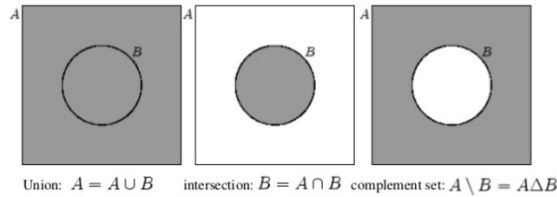
Summary:

- The complement of a set A (denoted A^c) is the set of all elements of S that do not belong to A . In terms of events, this is when event A did not happen.
- The intersection of A and B , denoted $A \cap B$, is the set of all elements that belong to *both* A and B .

- The union of A and B , denoted $A \cup B$, is the set of all elements that belong to *either* A or B .
- A, B disjoint/mutually exclusive iff $A \cap B = \emptyset$
- Unions of multiple sets $\bigcup_{i=1}^n A_i$ is similar to summation notation $\sum_{i=1}^4 x = x + x + x + x$.
- Intersections of multiple sets $\bigcap_{i=1}^n A_i$ is similar to the product operator $\prod_{i=1}^4 x = x^4$.



If $B \subset A$, some of the above diagrams are identical to one another:



Some important ideas, where A, B are generic sets and $C \subset A$:

- $A \setminus \emptyset = A$
- $A \cup \emptyset = A$
- $A \cap \emptyset = \emptyset$
- $A \cup B = B \cup A$
- $A \cap B = B \cap A$
- But it is **not** generally true that $A \setminus B = B \setminus A$
- $C \setminus A = \emptyset$, whereas $A \setminus C \neq \emptyset$
- $A \cup C = A$
- $A \cap A^c = \emptyset$

3.6 Cartesian Product

Definition: The Cartesian product of any two sets is the set containing all possible ordered pairs (a, b) where $a \in A$ and $b \in B$

$\{(3, 3); (3, elephant); (3, Q); (3, purple); (apple, 3); (apple, elephant); (apple, Q); (apple, purple); (Q, 3); (Q, elephant); (Q, Q); (Q, purple); (42, 3); (42, elephant); (42, Q); (42, purple), \}$

$$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$$

3.7 Interval Notation

Closed Set:

$$[0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$$

Open Set:

$$(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$$

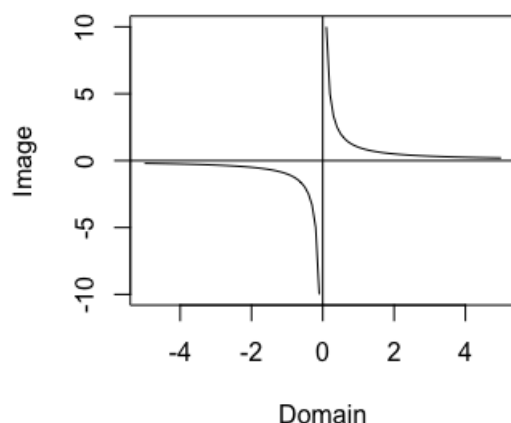
3.8 Practice questions

Find the sets that match each of the following statements, where $X = \{1, 2, 3\}$, $Y = \{A, B, 1, 2\}$, and $Z = \{B, C, 3, 4\}$, and Ω (the set containing every item in the space) is $\Omega = \{1, 2, 3, 4, A, B, C\}$.

- $X \cup Y = \{1, 2, 3, A, B\}$
- $X \cap Z = \{3\}$ NOTE: it is not just 3. It is $\{3\}$.
- $(X \cup Y) \cap Z = \{1, 2, 3, A, B\} \cap \{B, C, 3, 4\} = \{3, B\}$
- $\Omega \cup \emptyset = \{1, 2, 3, 4, A, B, C\}$
- $\Omega \cup X = \{1, 2, 3, 4, A, B, C\}$
- $(X \cup Y) \cap \emptyset = \emptyset$
- $(X \cup Y \cup Z)^c = \emptyset$
- $\emptyset^c = \{1, 2, 3, 4, A, B, C\}$
- $(X \cup Y) \setminus Z = \{1, 2, 3, A, B\} \setminus \{B, C, 3, 4\} = \{1, 2, A\}$

4 Derivatives

4.1 Limits



Before we can do calculus, we need one more concept in our pockets: the **limit**. The limit of a function is a value that its output approaches, but may never reach, as its input gets closer and closer to some value. Consider the function $y = \frac{1}{x}$, graphed above. As x increases, y approaches 0, but will never reach it. This suggests the following idea:

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

English: “The limit of the function as x approaches infinity is zero.”

We also say

English: “As x becomes arbitrarily close to infinity, y becomes arbitrarily close to zero.”

There’s a formal definition of what “arbitrarily close” means in this context, and that will be a major topic that you’ll cover in the next few weeks in POLSCI 598.

4.2 Sequences

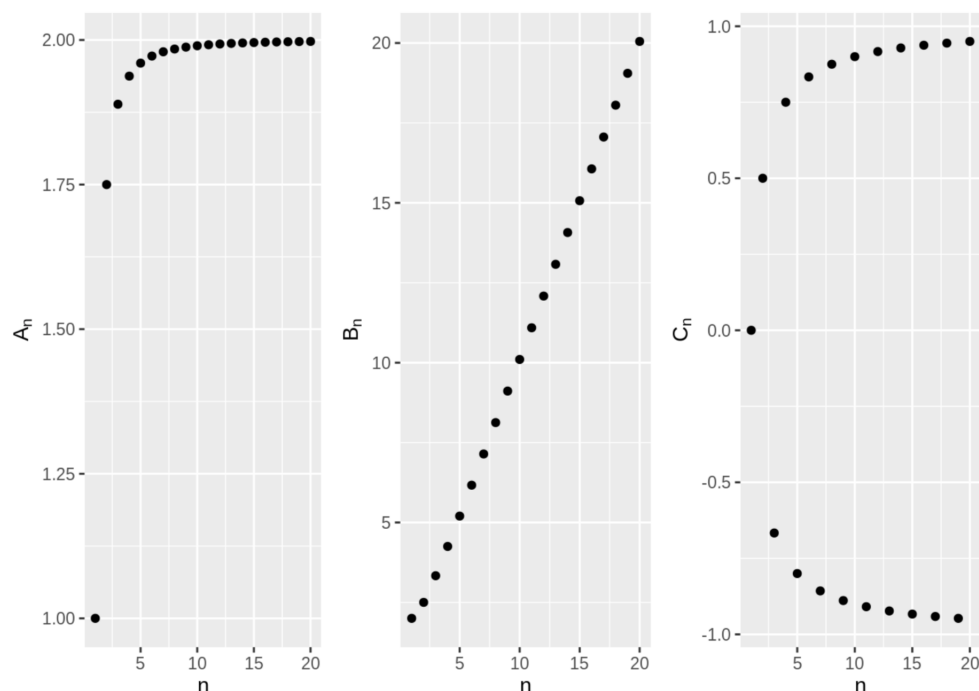
We need a couple of steps that will especially set up limit theorems in probability. First we will introduce a “sequence”, then we will think about the limit of a sequence, then we will think about the limit of a function.

A **sequence** of real numbers $X = \{x_1, x_2, x_3, \dots, x_n\}$ is an ordered set of real numbers, where x_1 is the first term in the sequence and x_n is the n th term. There may be as many elements of a sequence as there are natural numbers, so we could have infinitely long sequences of numbers. We can also write the sequence as $\{x_n\}_{n=1}^{\infty}$, where the subscript and superscript are read together as “from 1 to infinity.”

Example: How do these sequences behave? **Question:** Plug 1, 2, 3, 4, and 5 into each of the following to see what the first five elements in these sequences look like:

1. $\{a_n\} = \{2 - \frac{1}{n^2}\}$
2. $\{b_n\} = \{\frac{n^2+1}{n}\}$
3. $\{c_n\} = \{(-1)^n(1 - \frac{1}{n})\}$

The important thing is to get a sense of how these numbers are going to change. Example 1's numbers seem to come closer and closer to 2, but will it ever surpass 2? Example 2's numbers are also increasing each time, but will it hit a limit? What is the pattern in Example 3? Graphing helps you make this point more clearly. See the sequence of $n = 1, \dots, 20$ for each of the three examples in the figure below.



4.3 The Limit of a Sequence

The points in a sequence may or may not converge to a limit as $n \rightarrow \infty$.

1. Example 1 converges to a limit.
2. Example 2 increases without bound.
3. Example 3 does not converge, and also does not increase without bound, but continues diverging farther and farther apart

Definition: The sequence $\{y_n\}$ has the limit \mathbf{L}

$$\lim_{n \rightarrow \infty} y_n = \mathbf{L}$$

if for any $\epsilon > 0$, there is an integer N such that $|y_n - \mathbf{L}| < \epsilon$ for each $n > N$.

If \mathbf{L} is a limit of $\{y_n\}$, then we say that $\{y_n\}$ **converges to \mathbf{L}** . The opposite of converging is diverging.

A sequence is:

- Monotonically Increasing if $y_{n+1} > y_n \quad \forall n$
- Monotonically Decreasing if $y_{n+1} < y_n \quad \forall n$

A limit is unique. If $\{y_n\}$ converges, then the limit \mathbf{L} is unique.

A sum of converging sequences converges to the sum of their limits. In fact, letting $\lim_{n \rightarrow \infty} y_n = y$ and $\lim_{n \rightarrow \infty} z_n = z$, with $k, l \in \mathbb{R}$ and $z \neq 0$,

1. $\lim_{n \rightarrow \infty} [ky_n + lz_n] = ky + lz$ (we can move constants out of limits)
2. $\lim_{n \rightarrow \infty} y_n z_n = yz$
3. $\lim_{n \rightarrow \infty} \frac{y_n}{z_n} = \frac{y}{z}$

Example: (Simplifying a Fraction into Sums) Find the limit of

$$\lim_{n \rightarrow \infty} \frac{n+3}{n}$$

At first glance, $n+3$ and n both grow to ∞ , so it looks like we need to divide infinity by infinity. However, we can express this fraction as a sum, and then the limits apply separately:

$$\lim_{n \rightarrow \infty} \frac{n+3}{n} = \lim_{n \rightarrow \infty} \frac{n(1 + \frac{3}{n})}{n}$$

$$\lim_{n \rightarrow \infty} \frac{n+3}{n} = \lim_{n \rightarrow \infty} \left(1 + \frac{3}{n}\right)$$

$$\lim_{n \rightarrow \infty} \frac{n+3}{n} = \underbrace{\lim_{n \rightarrow \infty} 1}_1 + \underbrace{\lim_{n \rightarrow \infty} \left(\frac{3}{n}\right)}_0$$

$$\lim_{n \rightarrow \infty} \frac{n+3}{n} = 1$$

The intuitive key is to notice whether one part of the fraction grows “faster” (earlier in n) than another. If the denominator grows faster to infinity than the numerator, then the fraction will converge to 0, even if the numerator will also increase to infinity. This will become extremely important when you learn about estimators in POLSCI 699 semester after next.

Another good example is on page 181 of Jeff Gill’s 2006 book.

4.4 Limits of a Function

We have now covered functions and limits of sequences, so let’s combine the two topics.

A function f is a compact representation of some behaviour we care about. Like for sequences, we often want to know if $f(x)$ approaches some number \mathbf{L} as its independent variable x moves to some number c (which is usually 0 or $\pm\infty$). If it does, we say that the limit of $f(x)$, as x approaches c , is \mathbf{L} , denoted $\lim f(x) = \mathbf{L}$. Notice now we’re not just permitting x to approach ∞ , we can ask if the function has a limit at any arbitrary value of x .

For a limit \mathbf{L} to exist, the function $f(x)$ must approach \mathbf{L} from both the left (increasing x values) and the right (decreasing x values).

Properties: Let f and g be functions with $\lim_{x \rightarrow c} f(x) = k$ and $\lim_{x \rightarrow c} g(x) = l$, and r a constant.

1. $\lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x)$
2. $\lim_{x \rightarrow c} r f(x) = r \lim_{x \rightarrow c} f(x)$
3. $\lim_{x \rightarrow c} f(x) g(x) = [\lim_{x \rightarrow c} f(x)] \cdot [\lim_{x \rightarrow c} g(x)]$

$$4. \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}, \text{ provided } l \neq 0.$$

Question: (Do first together, and then everyone solve the rest on their own): Find the following limits, where k, c are constants:

1. $\lim_{x \rightarrow c} k$, **Answer:** k
2. $\lim_{x \rightarrow c} x$, **Answer:** c
3. $\lim_{x \rightarrow 2} (2x - 3)$, **Answer:** $= 2 \lim_{x \rightarrow 2} x - \lim_{x \rightarrow 2} 3 = 1$
4. $\lim_{x \rightarrow c} x^n$, **Answer:** $= \lim_{x \rightarrow c} x \dots \lim_{x \rightarrow c} x = \underbrace{c \dots c}_{c \text{ times}} = c^n$

4.5 Slope and linearity

Linear functions have the form, with $m, b \in \mathbb{R}$,

$$f(x) = mx + b$$

A function's **slope** is how much f changes when we increase x (i.e. "rise over run"). So for some fixed point x_0 and a change h , (draw it)

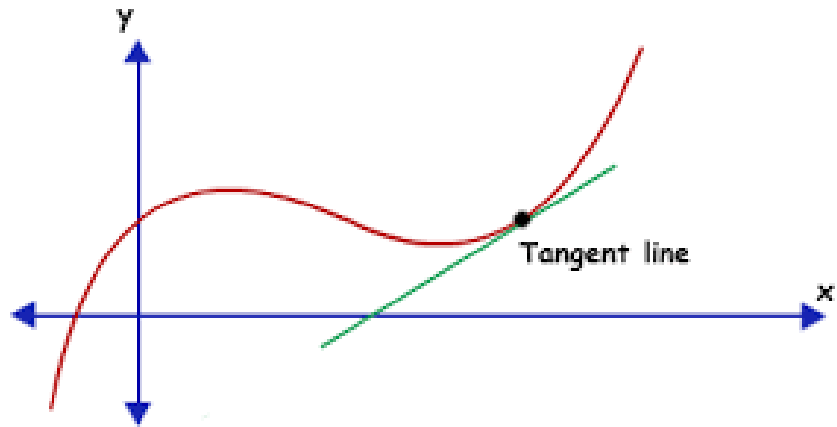
$$\frac{\text{rise}}{\text{run}} = \frac{f(x_0 + h) - f(x_0)}{h} = \frac{m(x_0 + h) + b - (mx_0 + b)}{h} = \frac{mh}{h} = m$$

Linear functions are great! Their slope is constant, always equal to the coefficient m . But what if the function isn't a line? Then the slope changes depending on the value of x , and we have to stop at

$$\frac{\text{rise}}{\text{run}} = \frac{f(x_0 + h) - f(x_0)}{h}$$

4.6 Tangent Lines

Our strategy for computing the slope of a curve? Draw a line with the same slope that intersects the curve. This is called a **tangent line**, and fortunately for us, we already know how to compute the slopes of lines!

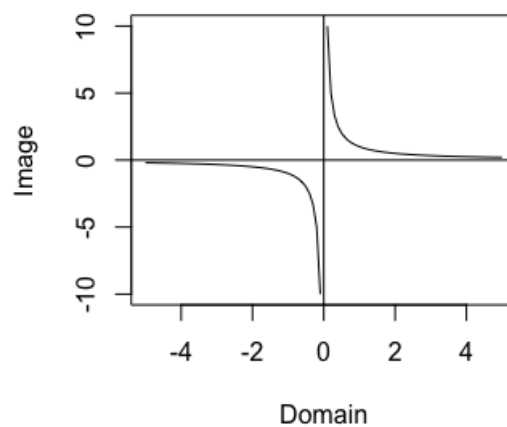


4.7 Asymptotes

Similar to the idea of tangent lines, an asymptote to a curve is a line that approaches the curve arbitrarily closely, so the distance between the line and the curve tends to zero but never reaches zero.

Question: Looking at the plot can you identify the asymptotes to the function

$$f(x) = \frac{1}{x}$$



4.8 Derivatives

The derivative of a function (at a point x_0) is the slope of the tangent line at that point. Or, in math:

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

The derivative of f at x is its rate of change at x : how much $f(x)$ changes with a change in x . The rate of change is a fraction – rise over run – but because not all lines are straight and the rise over run formula will give us different values depending on the range we examine, we need to take a limit.

Definition: Let f be a function whose domain includes an open interval containing the point x . The derivative of f at x is given by

$$\frac{d}{dx}f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{(x+h) - x} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

If $f'(x)$ exists at a point x_0 , then f is said to be **differentiable** at x_0 . That also implies that $f(x)$ is continuous at x_0 .

A walkthrough that's pitched similarly to math camp is around page 181 of Gill's book.

4.9 Some Notation

Several different ways to denote the slope of a function $y = f(x)$ at a given point.

Leibniz's notation:

$$\frac{dx}{dy}$$

Lagrange's notation:

$$f'(x)$$

Newton's notation:

$$\dot{f}(x)$$

We read Leibniz's notation notation as “the derivative of x with respect to y ”.

For functions of one variable, Leibniz's notation is a lot of extra symbols, so you'll probably most often encounter Lagrange's notation. But for multivariate functions, Leibniz's notation is essential, so it's important to be able to understand both of them? Newton's notation is very common overall but pretty rare in political science, because it's conventionally reserved for derivatives with respect to time. See [Wikipedia](#) for more information.

4.10 Properties of Derivatives

Suppose that f and g are differentiable at x and that α is a constant. Then the functions $f + g$, αf , fg , $\frac{f}{g}$ (provided $g(x) \neq 0$) are also differentiable at x .

4.10.1 Constant rule

$$[kf(x)]' = kf'(x)$$

4.10.2 Derivative of a constant

$$\frac{d}{dx}(c) = 0$$

4.10.3 Sum rule

$$[f(x) \pm g(x)]' = f'(x) \pm g'(x)$$

4.10.4 Product rule

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

4.10.5 Quotient rule

$$\left[\frac{f(x)}{g(x)} \right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

4.10.6 Power Rule

$$(x^k)' = kx^{k-1}$$

This last rule is the real workhorse of derivatives, so let's check an example of it.

$$f(x) = x^3$$

Using the power rule,

$$f'(x) = 3 \cdot x^{3-1}$$

$$f'(x) = 3x^2$$

Now, by our definition of a derivative, the slope of the function at a point x_0 is equal to:

$$\begin{aligned}
f'(x_0) &= \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \\
&= \lim_{h \rightarrow 0} \frac{(x_0 + h)^3 - (x_0)^3}{h} \\
&= \lim_{h \rightarrow 0} \frac{(x_0 + h)(x_0 + h)(x_0 + h) - x_0^3}{h} \\
&= \lim_{h \rightarrow 0} \frac{(x_0^2 + hx_0 + hx_0 + h^2)(x_0 + h) - x_0^3}{h} \\
&= \lim_{h \rightarrow 0} \frac{(x_0^2 + 2hx_0 + h^2)(x_0 + h) - x_0^3}{h} \\
&= \lim_{h \rightarrow 0} \frac{x_0^3 + 2hx_0^2 + h^2x_0 + hx_0^2 + 2h^2x_0 + h^3 - x_0^3}{h} \\
&= \lim_{h \rightarrow 0} \frac{x_0^3 + 3x_0^2h + 3x_0h^2 + h^3 - x_0^3}{h} \\
&= \lim_{h \rightarrow 0} \frac{3x_0^2h + 3x_0h^2 + h^3}{h} \\
&= \lim_{h \rightarrow 0} (3x_0^2 + 3x_0h + h^2) \\
&= 3x_0^2
\end{aligned}$$

It worked for $k = 3$. But obviously we do not want to have to do that every single time we want to take a derivative, so the power rule is crucial. Thankfully we can prove that it holds for any $k \in \mathbb{N}$.

In class we also did a second example, with $f(x) = 10x^2$. By applying the power rule (multiply it by the exponent, then subtract 1 from the exponent), we get the result $f'(x) = 20x$. Now using the definition of a derivative,

$$f'(x) = \lim_{h \rightarrow 0} \frac{10(x+h)^2 - 10(x)^2}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{10(x+h)(x+h) - 10x^2}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{10(x^2 + 2xh + h^2) - 10x^2}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{10x^2 + 20xh + 10h^2 - 10x^2}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{20xh + 10h^2}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} (20x + 10h)$$

$$f'(x) = 20x$$

So in this case too, the two methods agree.

4.11 Some common derivatives

1. $\frac{d}{dx}(x) = 1$ (Power rule)
2. $\frac{d}{dx}(e^x) = e^x$ (We'll discuss this in 598)
3. $\frac{d}{dx} \frac{1}{x} = -\frac{1}{x^2}$ (Quotient rule or power rule)
4. $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$, $x > 0$ (We'll discuss this in 598)

4.12 Derivative examples and practice

There are two good reminder sheets for differentiation at [Derivative Rules, short](#) and [Derivative Rules, long](#).

Examples: Find the derivatives of the following functions (2 together, 3 on your own):

1. $\frac{d}{dx}(3x - 1) = \frac{d}{dx}(3x) - \frac{d}{dx}(1)$

$$\frac{d}{dx}(3x - 1) = 3$$

What does this mean again? It says the instantaneous rate of change of the function at any point x is 3. Huh? The rate of change of this function is 3 everywhere? **Draw it** – does that make sense as a derivative?

2. $\frac{d}{dx}(5x + 2) = \frac{d}{dx}(5x) - \frac{d}{dx}(2)$

$$\frac{d}{dx}(5x + 2) = 5$$

3. $f(x) = x^2 - x + 3$

$$f'(x) = (x^2 - x + 3)' = (x^2)' - (x)' + (3)' = 2x - 1 + 0 = 2x - 1$$

4. Try both terms individually before trying this one: $f(x) = (3x^2 - 4x + 1)(8x + 7)$

$$\begin{aligned} f'(x) &= ((3x^2 - 4x + 1)(8x + 7))' = (3x^2 - 4x + 1)' \cdot (8x + 7) + (3x^2 - 4x + 1) \cdot (8x + 7)' \\ &= (6x - 4 + 0)(8x + 7) + 8(3x^2 - 4x + 1) = 48x^2 + 42x - 32x - 28 + 24x^2 - 32x + 8 = \\ &= 72x^2 - 22x - 20 \end{aligned}$$

5. $f(x) = \frac{3x - 1}{x}$

By the quotient rule:

$$f'(x) = \left(\frac{3x - 1}{x} \right)' = \frac{(3x - 1)' \cdot x - (3x - 1) \cdot x'}{x^2} = \frac{3x - 3x + 1}{x^2} = \frac{1}{x^2}$$

By the product rule:

$$\begin{aligned} f'(x) &= \left((3x - 1)(x^{-1}) \right)' = (3x - 1)'(x^{-1}) + (3x - 1)(x^{-1})' = 3x^{-1} - (3x - 1)(x^{-2}) = \\ &= \frac{3}{x} - \frac{3}{x} + \frac{1}{x^2} = \frac{1}{x^2} \end{aligned}$$

6. $f(x) = (x^3)(2x^4)$

$$f'(x) = ((x^3)(2x^4))' = (x^3)' \cdot 2x^4 + x^3 \cdot (2x^4)' = 3x^2 \cdot 2x^4 + x^3 \cdot 8x^3 = 6x^6 + 8x^6 = 14x^6$$

7. $f(x) = \frac{1}{x^2}$

$$f'(x) = -2x^{-3}$$

4.13 Composite Functions and the Chain Rule

Despite all the rules we have covered so far, many functions will still not fit neatly in each case immediately. Instead, they will be functions of functions. For example, the difference between $x^2 + 1^2$ and $(x^2 + 1)^2$ means that the sum rule can be easily applied to the former, while it is actually not obvious what to do with the latter.

As mentioned in the previous sections, **composite functions** are formed by substituting one function into another and are denoted by

$$(f \circ g)(x) = f(g(x))$$

Chain Rule: Let $y = (f \circ g)(x) = f(g(x))$. The derivative of y with respect to x is

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

The chain rule can be thought of as the derivative of the “outside” times the derivative of the “inside”, remembering that the derivative of the outside function evaluated at the value of the inside function. It can also be written as

$$\frac{dy}{dx} = \frac{dy}{dg(x)} \cdot \frac{dg(x)}{dx}$$

This expression does not imply that the $dg(x)$ s cancel out, as in fractions. They are part of the derivative notation and you cannot separate them out or cancel them.

Example (composite exponent): Find $f'(x)$ for $f(x) = (3x^2 + 5x - 7)^6$.

We have a generalized power rule: If $f(x) = [g(x)]^p$ for any rational number p ,

$$f'(x) = p[g(x)]^{p-1}g'(x)$$

$$6 \cdot (3x^2 + 5x - 7)^5 \cdot (6x + 5)$$

Examples:

Find the derivative for the following (2 together, 3 on your own):

$$1. f(x) = (6x^2 + 7x)^4$$

$$f'(x) = 4(6x^2 + 7x)^3(12x + 7) = 4(12x + 7)(6x^2 + 7x)^3$$

$$2. g(t) = (4t^2 - 3t + 2)^{-2}$$

$$g'(t) = -2(4t^2 - 3t + 2)^{-3}(8t - 3) = -2(8t - 3)(4t^2 - 3t + 2)^{-3}$$

$$3. f(x) = (3x + 1)^2$$

$$f'(x) = 2(3x + 1)(3) = 6(3x + 1)$$

$$4. f(x) = \sqrt{13x^2 - 5x + 8}$$

$$f'(x) = ((13x^2 - 5x + 8)^{\frac{1}{2}})' = \frac{1}{2}(13x^2 - 5x + 8)^{\frac{1}{2}-1}(13x^2 - 5x + 8)'$$

$$= \frac{1}{2}(13x^2 - 5x + 8)^{-\frac{1}{2}}(26x - 5)$$

$$= \frac{26x - 5}{2(13x^2 - 5x + 8)^{\frac{1}{2}}}$$

$$= \frac{26x - 5}{2\sqrt{13x^2 - 5x + 8}}$$

$$5. \frac{d}{dx} \left(\frac{3x-1}{5x+2} \right) = \frac{d}{dx} (3x - 1)(5x + 2)^{-1}$$

$$= (3x - 1)'(5x + 2)^{-1} + (3x - 1)((5x + 2)^{-1})'$$

$$= (3)(5x + 2)^{-1} - 5(3x - 1)(5x + 2)^{-2}$$

$$= \frac{3}{5x+2} - \frac{5(3x-1)}{(5x+2)^2}$$

$$= \frac{3(5x+2)}{(5x+2)^2} - \frac{5(3x-1)}{(5x+2)^2}$$

$$= \frac{15x+6-15x+5}{(5x+2)^2}$$

$$= \frac{11}{(5x+2)^2}$$

4.14 Derivatives of natural logs and the exponent

e is the constant such that

$$(e^x)' = e^x$$

We'll talk more about this somewhat mistifying claim in 598.

$\ln(x)$ is continuous and differentiable, and its first derivative is

$$\ln(x)' = \frac{1}{x}$$

Also, when these are composite functions, it follows by the generalized power rule that

$$(e^{g(x)})' = e^{g(x)} \cdot g'(x)$$

$$(\ln g(x))' = \frac{g'(x)}{g(x)}$$

if $g(x) > 0$.

Derivative of $e^{f(x)}$:

1. Derivative of e^x is itself: $\frac{d}{dx}e^x = e^x$
2. Same thing if there were a constant in front: $\frac{d}{dx}ae^x = ae^x$
3. Same thing no matter how many derivatives there are in front: $\frac{d}{dx}\left(\frac{d}{dx}(ae^x)\right) = ae^x$
4. Chain Rule: When the exponent is a function of x , take the derivative of that function and multiply $\frac{d}{dx}e^{g(x)} = e^{g(x)}g'(x)$.

Examples:

Find the derivatives of the following:

1. $f(x) = e^{-3x} = -3e^{-3x}$
2. $f(x) = e^{x^2} = 2xe^{x^2}$

Derivative of logarithms:

1. $\frac{d}{dx} \log x = \frac{1}{x}$
2. Exponents become multiplicative constants: $\frac{d}{dx} \log x^k = \frac{d}{dx} k \log x = \frac{k}{x}$
3. Chain rule: $\frac{d}{dx} \log u(x) = \frac{u'(x)}{u(x)}$
4. For any positive base b , $\frac{d}{dx} b^x = (\log b)(b^x)$.

Examples:

Find the derivatives of the following:

1. $y = \ln(x^2 + 9)$
Let $u(x) = x^2 + 9$. Then $u'(x) = 2x$ and $\frac{dy}{dx} = \frac{u'(x)}{u(x)} = \frac{2x}{(x^2+9)}$
2. $f(y) = \ln(1 - 5y^2 + y^3)$
 $f'(y) = \frac{1}{1-5y^2+y^3}(-10y + 3y^2) = \frac{-10y+3y^2}{1-5y^2+y^3}$
3. $y = \ln(\ln x)$
Let $u(x) = \ln x$. Then $u'(x) = \frac{1}{x}$ and $\frac{dy}{dx} = \frac{1}{(x \ln x)}$
4. $y = (\ln x)^2$
Using the generalized power rule: $\frac{dy}{dx} = \frac{2 \ln x}{x}$
5. $y = \ln e^x$
Let $u(x) = e^x$. Then $u'(x) = e^x$ and $\frac{dy}{dx} = \frac{u'(x)}{u(x)} = \frac{e^x}{e^x} = 1$.
Alternatively, just notice $\ln(e^x) = x$.

4.15 Second Derivatives

What if we take the derivative of a derivative?

$$f(x) = x^3$$

$$f'(x) = 3x^2$$

$$f''(x) = 6x$$

The **second derivative** gives us the slope of a function's derivative (i.e. the rate of change of the rate of change or how fast the rate of change is changing). In Leibniz notation, the second derivative is denoted $\frac{d^2y}{dx^2}$.

We had mentioned that if a function is differentiable at a given point, then it must be continuous. Further, if $f'(x)$ is itself continuous, then $f(x)$ is called continuously differentiable. We will use this very heavily in 598. A function that is continuously differentiable infinitely is called “smooth”. Some examples: $f(x) = x^2$, $f(x) = e^x$.

5 Integrals

5.1 Antiderivatives

If we can take the derivative of a function, maybe we can reverse the process too, and find the function F for which f is its derivative. An **antiderivative** answers the question: what is the function $F(x)$ that has the derivative $f(x)$?

Example:

$$\begin{aligned}f(x) &= 3x^2 \\F(x) &= x^3 + c\end{aligned}$$

Notice that we had to add a value c , called the “constant of integration”, because a constant could have been eliminated by the process of taking a derivative.

Example: What might we have taken the derivative of to obtain the following function?

1. $f(x) = 9x^8 + 2x + 1$ **Answer:** $F(x) = x^9 + x^2 + x + c$

We know from derivatives how to manipulate $F(x)$ to get $f(x)$. But what procedure do we follow to get $F(x)$ back from $f(x)$? For that, we will need a new symbol, which we will call indefinite integration.

5.2 Indefinite Integral

The indefinite integral of $f(x)$ is written

$$F(x) = \int f(x)dx$$

and is equal to the antiderivative of $f(x)$. While there is only a single derivative for any function, there are multiple antiderivatives: one for any arbitrary constant c . For example, $\frac{1}{3}x^3 - 4x$ and $\frac{1}{3}x^3 - 4x + 1$ are both antiderivatives of $f(x) = (x^2 - 4)$.

5.3 Common Rules of Integration

Let’s just start by setting out the rules of integration that we know will have to be true in order for them to reverse the process of taking a derivative.

1. Constants: $\int af(x)dx = a \int f(x)dx$
2. Sums: $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

3. Reverse Power-rule: $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$
4. Similar rule for e : $\int e^x dx = e^x + c$
5. Recall the derivative of $\ln(x)$ is $\frac{1}{x}$, and so: $\int \frac{1}{x} dx = \ln|x| + c$
6. Reverse chain rule: $\int e^{f(x)} f'(x) dx = e^{f(x)} + c$
7. More reverse chain rule: $\int [f(x)]^n f'(x) dx = \frac{1}{n+1} [f(x)]^{n+1} + c$
8. Somehow even more reverse chain rule: $\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$

Examples:

1. $\int 3x^2 dx$
 Take the constant out: $= 3 \cdot \int x^2 dx$
 Apply the power rule: $= 3 \cdot \frac{x^{2+1}}{2+1} = x^3$
 Add a constant to the solution: $= x^3 + c$
2. $\int (2x + 1) dx$
 Apply the sum rule: $= \int 2x dx + \int 1 dx$
 Take the constant out: $2 \cdot \int x dx + \int 1 dx$
 Apply the power rule: $= 2 \cdot \frac{x^{1+1}}{1+1} + c_1 + \int 1 dx$
 Integral of a constant is just $x + c$: $= 2 \cdot \frac{x^2}{2} + c_1 + x + c_2$
 Simplify: $= x^2 + x + c$
3. $\int (5x + 5) dx = \int 5x dx + \int 5 dx$
 $= \frac{5}{2}x^2 + 5x + c$
4. $\int (6x + x^2) dx = \int 5x dx + \int x^2 dx$
 $= \frac{6}{2}x^2 + \frac{1}{3}x^3 + c$
 $= 3x^2 + \frac{1}{3}x^3 + c$
5. $\int (12x^3 + 9x^2) dx = \int 12x^3 dx + \int 9x^2 dx$
 $\int 4x^4 dx + 3x^3 + c$

$$6. \int (10y)dy = 5y^2 + c$$

$$7. \int (10y)dx = 10xy + c$$

Now everyone try the following questions on your own. First, take the derivative of the given function. Then, apply the integral to the derivative. By applying the derivative and then the integral, you want to show that you recover the original function:

$$1. f(x) = x^3 + 5x - 4$$

$$f'(x) = 3x^2 + 5$$

$$\int f'(x)dx = \int (3x^2 + 5)dx$$

$$\int f'(x)dx = x^3 + 5x + c$$

$$2. f(x) = 8x^4 + 2x^3 - 4x + 1$$

$$f'(x) = 32x^3 + 6x^2 - 4$$

$$\int f'(x)dx = \frac{32}{4}x^4 + \frac{6}{3}x^3 - 4x + c$$

$$\int f'(x)dx = 8x^4 + 2x^3 - 4x + c$$

$$3. f(y) = 7y^{\frac{3}{2}} - 9$$

$$f'(y) = \frac{21}{2}y^{\frac{1}{2}}$$

$$\int f'(y)dx = \frac{2}{3}\frac{21}{2}y^{\frac{3}{2}} + c$$

$$\int f'(y)dx = \frac{21}{3}y^{\frac{3}{2}} + c$$

$$\int f'(y)dx = 7y^{\frac{3}{2}} + c$$

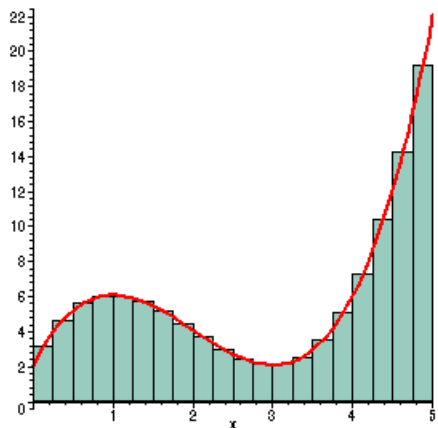
$$4. f(x) = 2z$$

5. I dare you to pick any polynomial and try it!

5.4 Integrals

We built up derivatives as the slope of the line tangent to a curve. Then we introduced this thing called the antiderivative, and we called it the “integral”. So what exactly does this function mean, apart from undoing derivatives?

Suppose we want to compute the area beneath an arbitrarily squiggly function. A natural idea is the **Riemann Sum**: we draw a bunch of rectangles under the curve, and just add up the total area of those rectangles.



—
Riemann Sum

As we decrease the width of the rectangles, the sum of their areas approaches the area under the curve. As the limit of the width of each rectangle approaches zero, the Riemann Sum of a curve from $x = a$ to $x = b$ approaches the **definite integral**, so:

$$\lim_{w \rightarrow 0} \sum_{i=1}^N f(x_i) \cdot w = \int_a^b f(x) dx$$

5.5 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus formalises what we’ve just seen: that integrals are antiderivatives. Let the function f be bounded on $[a, b]$ and continuous on (a, b) , and F any function that is continuous on $[a, b]$ such that $F'(x) = f(x)$ on (a, b) . Then

$$\int_a^b f(x) dx = F(b) - F(a)$$

So the procedure to calculate a simple definite integral $\int_a^b f(x) dx$ is then a) finding the indefinite integral $F(x)$ and b) evaluating $F(b) - F(a)$.

English: Integrals and derivatives are reverse operations. Finding the area under a curve is equivalent to taking the antiderivative.

Note: For definite integrals, we know exactly the value! So we do not have to add a constant. We only add a constant to indefinite integrals.

5.6 Common Rules for Definite Integrals

Having seen the area interpretation of the definite integral, we can state some more rules.

1. Not all intervals have area. For example: $\int_a^a f(x)dx = 0$
2. Reversing the bounds changes the sign of the integral: $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. Sums can be separated into their own integrals: $\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$
4. Areas can be combined as long as limits are linked: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$
5. $\int_a^b 1dx = b - a$
6. $\int_a^b xdx = \frac{1}{2}(b^2 - a^2)$
7. Integral of a constant $k \in \mathbb{R}$: $\int_a^b kdx = k(b - a)$

Examples:

1. Example of property 1: $\int_1^1 3x^2dx = 0$
2. Example of property 2: $\int_0^1 x^3dx = \left[\frac{1}{4}x^4\right]_0^1$

$$= \frac{1}{4}(1)^4 - \frac{1}{4}(0)^4$$

$$= \frac{1}{4}$$

$$\text{In contrast, } \int_1^0 x^3dx = \left[\frac{1}{4}x^4\right]_1^0$$

$$= \frac{1}{4}(0)^4 - \frac{1}{4}(1)^4$$

$$= -\frac{1}{4}$$

$$\begin{aligned}
3. \text{ Example of property 3: } \int_1^3 (4x^2 + 2x^3) dx &= \int_1^3 (4x^2) dx + \int_1^3 (2x^3) dx \\
&= 4 \int_1^3 (x^2) dx + 2 \int_1^3 (x^3) dx \\
&= 4 \left[\frac{1}{3} x^3 \right]_1^3 + 2 \left[\frac{1}{4} x^4 \right]_1^3 dx \\
&= \frac{4}{3} (3^3 - 1^3) + \frac{1}{2} (3^4 - 1^4) \\
&= \frac{4}{3} \cdot 26 + \frac{1}{2} \cdot 80 \\
&= \frac{224}{3}
\end{aligned}$$

$$4. \int_0^4 (2x + 1) dx = 20$$

5. Find the area under the function $f(x) = 2x$ between $x = 0$ and $x = 3$. Use both the FTC and the formula for the area of a triangle, $A = \frac{1}{2}w \cdot h$ **Answer:** 9

$$\begin{aligned}
6. \int_1^3 3x^2 dx &= [x^3]_1^3 \\
&= 26
\end{aligned}$$

More Integration Rules: [Integration Rules \(with bonus derivative rules on page 1!\)](#).

5.7 Integration by Substitution

Our operation for computing a definite integral was to first find the indefinite integral, and then plug in the bounds and subtract.

However, sometimes the integrand (the thing that we are trying to take an integral of) doesn't appear integrable using the rules we've discussed for antiderivatives. In many of these cases we can get the result using integration by substitution, which is analogous to the Chain Rule, and is also called u -substitution:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du \text{ where } u = g(x)$$

So, $F[u(x)]$ is the antiderivative of g . We can then write

$$\int g(x)dx = \int f[u(x)]u'(x)dx = \int \frac{d}{dx} F[u(x)]dx = F[u(x)] + c$$

To summarize, the procedure to determine the indefinite integral $\int g(x)dx$ by the method of substitution:

1. Identify some part of $g(x)$ that might be simplified by substituting in a single variable u .
2. Determine if $g(x)dx$ can be reformulated in terms of u and du .
3. Solve the indefinite integral.
4. Substitute back in for x

Substitution can also be used to calculate a definite integral. Using the same procedure as above,

$$\int_a^b g(x)dx = \int_c^d f(u)du = F(d) - F(c)$$

where $c = u(a)$ and $d = u(b)$.

How do you know when to use u-substitution? A **rule of thumb**, ask yourself what portion of the integrand has an inside function that is making the integral difficult to apply our integration rules to. If there is an inside function that's making things difficult, it's a good candidate to substitute out.

Another thing to keep an eye on is that, after we perform the substitution, every x in the integral (including the x in the symbol dx) needs to disappear, and the only variables left should be u 's (including a du).

Example 1: $\int 3(8y - 1)e^{4y^2 - y}dy$

Choose the substitution $u = 4y^2 - y$ so $du = (8y - 1)dy$.

We can factor out the constant so the integral becomes

$$\begin{aligned}\int 3(8y - 1)e^{4y^2 - y}dy &\Rightarrow 3 \int e^u du \\ &= 3e^u + c \\ &= 3e^{4y^2 - y} + c\end{aligned}$$

Example 2: $\int \frac{1}{1-2x}dx$

Choose the substitution $u = 1 - 2x$ so $du = -2dx$.

So the integral becomes

$$\begin{aligned}\int \frac{1}{u}(-\frac{1}{2}du) &= -\frac{1}{2} \int \frac{1}{u} du \\ &= -\frac{1}{2} \ln|u| + c \\ &= -\frac{1}{2} \ln|1 - 2x| + c\end{aligned}$$

The result of this example can be generalised: if we want to find $\int \frac{1}{ax+b} dx$, the substitution $u = ax + b$ leads to $\frac{1}{a} \int \frac{1}{u} du$ which equals $\frac{1}{a} \ln|ax + b| + c$. This means, for example, that when faced with an integral such as $\int \frac{1}{3x+7} dx$, we can immediately write down the answer as $\frac{1}{3} \ln|3x + 7| + c$.

5.8 Integration by Parts

We use integration by parts to integrate products of functions.

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

If we use some simple substitutions, this rule gets much easier to use. Substitute

$$\begin{array}{ll}u = f(x) & v = g(x) \\ du = f'(x)dx & dv = g'(x)dx\end{array}$$

Then,

$$\int u dv = uv - \int v du$$

Example: Evaluate $\int xe^x dx$. We want to separate x from e^x . So choose the labels

$$u = x$$

yielding

$$du = dx$$

and then we want

$$dv = e^x dx$$

in which case

$$v = e^x$$

Now, following the formula for integration by parts,

$$\int x e^x dx = \int u dv$$

$$\int x e^x dx = uv - \int v du$$

Plugging in u and v ,

$$\int x e^x dx = x e^x - \int e^x dx$$

$$\int x e^x dx = x e^x - e^x + c$$

Example: Evaluate $\int x e^{6x} dx$

Choose

$$u = x$$

$$du = dx$$

$$dv = e^{6x} dx$$

$$v = \int e^{6x} dx = \frac{1}{6} e^{6x}$$

Then the integral is

$$\begin{aligned} \int x e^{6x} dx &= \frac{x}{6} e^{6x} - \int \frac{1}{6} e^{6x} dx \\ &= \frac{x}{6} e^{6x} - \frac{1}{36} e^{6x} + c \end{aligned}$$

6 Linear Algebra

6.1 Vectors

Let's think of **vectors** in the following way: they are a data storage device that hold numbers in an ordered list. These lists come in two formats: row vectors and column vectors.

Example: $\mathbf{x} = [1 \ 3 \ 7]$

$$\mathbf{a} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\mathbf{b} = [x_1 \ x_2 \ \dots \ x_n]$$

Turning a row vector v into a column vector (taking the leftmost element and putting it on top, then the second leftmost element and putting it second from top, and so on), or vice versa, is called taking the **transpose** and is denoted \mathbf{v}' or \mathbf{v}^T . In the above example, $\mathbf{a} = \mathbf{b}'$.

It is very common to use an n -dimensional vector to associate one number with each of n dimensions in \mathbb{R}^n .

Example: Draw the vector $[2 \ 1]$ starting from $(0,0)$.

In contrast to vectors, **scalars** are individual numbers. So $x \in \mathbb{R}$ is a scalar, while $\mathbf{x} = [-1 \ \pi]$ is a vector, $x \in \mathbb{R}^2$.

Like with sets, we call the components of a vector **elements**.

Note: Now we've talked about two different list-like storage devices, namely vectors and sets. Let's talk for a moment about the differences between the two.

1. Unlike sets, the elements of a vector are always numbers (or variables that stand in for numbers). It's incoherent to talk about a vector that contains the element *elephant*, like our example of a set did.
2. This means that, unlike with sets, there's no such thing as "a vector of vectors".
3. Unlike with sets, the order of a vector really matters. It is absolutely fixed. So we can refer to the elements of vectors using subscripts. For the vector $\mathbf{x} = [-1 \ \pi]$, $x_1 = -1$

and $x_2 = \pi$.

Rejoice, for in a way vectors are much simpler than sets.

6.1.1 Vector Algebra

Vector addition (and **vector subtraction**) is simply the addition of the corresponding elements in two vectors. If two vectors, \mathbf{u} and \mathbf{v} , have the same dimension n (by which I mean, contain the same number of numbers — I thank an anonymous benefactor for suggesting the word “dimension”, which was a surprisingly tricky piece of terminology), they can be added together:

$$\mathbf{u} + \mathbf{v} = [u_1 + v_1 \quad u_2 + v_2 \quad \dots \quad u_n + v_n]$$

$$\mathbf{u} - \mathbf{v} = [u_1 - v_1 \quad u_2 - v_2 \quad \dots \quad u_n - v_n]$$

Example: If $\mathbf{u} = [1 \quad 2 \quad 3]$ and $\mathbf{v} = [3 \quad -1 \quad \pi]$ then

$$\mathbf{u} + \mathbf{v} = [1 \quad 2 \quad 3] + [3 \quad -1 \quad \pi]$$

$$\mathbf{u} + \mathbf{v} = [1 + 3 \quad 2 - 1 \quad 3 + \pi]$$

$$\mathbf{u} + \mathbf{v} = [4 \quad 1 \quad 3 + \pi]$$

Vector addition and vector subtraction are only defined for vectors of the same dimensionality. So we cannot calculate $[2 \quad 1] - [0 \quad 0 \quad 5]$.

We can also interpret vector addition geometrically: it's the sum of their extent in every dimension that they're defined on. (**Draw it** for $[2 \quad 1] + [3 \quad 1]$)

Scalar multiplication of vectors involves multiplying vectors by scalars. The product of a scalar c and vector \mathbf{v} is:

$$c\mathbf{v} = [cv_1 \quad cv_2 \quad \dots \quad cv_n]$$

Notice that vector subtraction can be defined using vector addition and scalar multiplication:

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v}$$

Questions: Let $\mathbf{a} = [1 \ 3 \ 7]$, $\mathbf{b} = [-5 \ 3 \ 2]$, and $c = 5$. Calculate the following:

- $5\mathbf{a}$ **Answer:** $[5 \ 15 \ 35]$
- $2\mathbf{a} + \mathbf{b}$ **Answer:** $[-3 \ 9 \ 12]$
- $-1\mathbf{a} - \mathbf{b}$ **Answer:** $[4 \ -6 \ -9]$

6.1.2 Dot products, lengths, distances

The **dot product** (also called the **inner product**) of two vectors is the sum of the products of their corresponding elements, so

$$\mathbf{a} \cdot \mathbf{b} = \sum_i a_i \cdot b_i$$

For column vectors, this is equal to $\mathbf{a}'\mathbf{b}$, recalling that $'$ indicates the transpose. We will see why this is true when we study matrix multiplication.

Again, you can see that the dot product of two vectors is only defined when those vectors have the same number of elements:

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n = \sum_{i=1}^n u_iv_i$$

We call two vectors \mathbf{u} and \mathbf{v} **orthogonal** (or **perpendicular**) if $\mathbf{u} \cdot \mathbf{v} = 0$

Example: To check the orthogonality of $[2 \ 3]$ and $[-3 \ 2]$, we just verify that their dot product is equal to zero. **Draw them.**

Question: Come up with two vectors of dimension 2 that have a dot product equal to zero. Now draw them. Can you see the reasoning behind the definition?

A **norm** is a function that maps vectors onto real numbers in a way that captures the idea of distance or length. You'll talk more about several different norms in 598, each of which tries to capture a different idea of length. The most common is the **Euclidean norm** or **2-norm**:

$$||\mathbf{x}||_2 = \left(\sum_{i=1}^m |x_i|^2 \right)^{\frac{1}{2}}$$

(Note the absolute value sign is there because there are norms which raise each index to an odd power)

Suppose $x = \begin{bmatrix} 4 & 3 \end{bmatrix}$. Then

$$||\mathbf{x}||_2 = \sqrt{4^2 + 3^2}$$

$$||\mathbf{x}||_2 = \sqrt{16 + 9}$$

$$||\mathbf{x}||_2 = \sqrt{25}$$

$$||\mathbf{x}||_2 = 5$$

Why does this capture an idea of distance? Remember the idea that each part of the vector represents a distance in a different dimension. (**Example:** draw out the vector in 2 dimensions and show it's the distance between the points on each axis).

In 2 dimensions this is just our old friend the Pythagorean theorem: $a^2 + b^2 = c^2$.

Another way of writing the 2-norm is as a dot product. Consider the vector $\mathbf{a} = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$. According to the 2-norm, we said that the length of the vector \mathbf{a} is

$$||\mathbf{a}||_2 = \sqrt{a_1^2 + a_2^2}$$

Now notice that this is exactly what you get by computing the following:

$$\sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1 \cdot a_1 + a_2 \cdot a_2}$$

$$\sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2}$$

$$\sqrt{\mathbf{a} \cdot \mathbf{a}} = \|\mathbf{a}\|_2$$

Example: $\mathbf{a} = [1 \ 3 \ 5]$. Length of $\mathbf{a} = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{1^2 + 3^2 + 5^2} = \sqrt{35}$.

To find the **Euclidean distance** between two vectors \mathbf{a} and \mathbf{b} , denoted $\|\mathbf{a} - \mathbf{b}\|$, we take the difference between them and then apply the 2-norm, so $\|\mathbf{a} - \mathbf{b}\| = \sqrt{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})}$.

Example: Find the distance between $\mathbf{u} = [1 \ 2]$ and $\mathbf{v} = [4 \ 5]$:

$$|\mathbf{u} - \mathbf{v}| = \|\mathbf{u} - \mathbf{v}\|_2$$

$$|\mathbf{u} - \mathbf{v}| = \sqrt{(1-4)^2 + (2-5)^2}$$

$$|\mathbf{u} - \mathbf{v}| = \sqrt{9+9}$$

$$|\mathbf{u} - \mathbf{v}| = \sqrt{18}$$

Example: Find the distance between $\mathbf{u} = [1 \ 2 \ 3]$ and $\mathbf{v} = [2 \ 4 \ 6]$:

$$|\mathbf{u} - \mathbf{v}| = \sqrt{(1-2)^2 + (2-4)^2 + (3-6)^2}$$

$$|\mathbf{u} - \mathbf{v}| = \sqrt{(-1)^2 + (-2)^2 + (-3)^2}$$

$$|\mathbf{u} - \mathbf{v}| = \sqrt{1+4+9}$$

$$|\mathbf{u} - \mathbf{v}| = \sqrt{14}$$

Practice questions: Let $\mathbf{a} = [2 \ 1 \ 2]$, $\mathbf{b} = [3 \ 4 \ 5]$. Calculate the following:

$$1. \mathbf{a} - \mathbf{b} = \begin{bmatrix} -1 & -3 & -3 \end{bmatrix}$$

$$2. \mathbf{a} \cdot \mathbf{b} = 6 + 4 + 10 = 20$$

Now let $\mathbf{u} = \begin{bmatrix} 7 & 1 & -5 & 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 9 & -3 & 2 & 8 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 1 & 13 & -7 & 2 & 15 \end{bmatrix}$, and $c = 2$. Calculate each of the following, or explain why they cannot be calculated:

$$1. \mathbf{u} - \mathbf{v} = \begin{bmatrix} -2 & 4 & -7 & -5 \end{bmatrix}$$

$$2. c\mathbf{w} = \begin{bmatrix} 2 & 26 & -14 & 4 & 30 \end{bmatrix}$$

$$3. \mathbf{u} \cdot \mathbf{v} = 63 - 3 - 10 + 24 = 74$$

$$4. \mathbf{w} \cdot \mathbf{v} = \text{undefined}$$

6.1.3 Linear combinations, span of a set

Consider $n \in \mathbb{N}$ vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. A vector \mathbf{y} is called a **linear combination** of the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ if $\exists n$ scalars c_1, c_2, \dots, c_n such that $\mathbf{y} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots + c_n\mathbf{x}_n$.

Example: $\begin{bmatrix} 9 & 13 & 17 \end{bmatrix}$ is a linear combination of the following three vectors: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$, and $\begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$. This is because $\begin{bmatrix} 9 & 13 & 17 \end{bmatrix} = (2) \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + (-1) \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} + (3) \begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$

Questions:

- Is $\mathbf{u} = \begin{bmatrix} 1 & 2 \end{bmatrix}$ a linear combination of $\mathbf{v} = \begin{bmatrix} 2 & 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 & -1 \end{bmatrix}$? **Answer:** Yes, $\mathbf{u} = \mathbf{v} - \mathbf{w}$.
- Is $\mathbf{u} = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix}$ a linear combination of $\mathbf{v} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 & 4 & 2 \end{bmatrix}$? **Answer:** No.
- Is $\mathbf{u} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ a linear combination of $\mathbf{v} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 & 0 & 100 \end{bmatrix}$? **Answer:** Yes, $\mathbf{u} = 2\mathbf{v} + 0\mathbf{w}$.
- Is $\mathbf{u} = \begin{bmatrix} \pi & \pi & 2\pi & 4\pi \end{bmatrix}$ a linear combination of $\mathbf{v} = \begin{bmatrix} \pi & \pi & 0 & 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 & 0 & \frac{1}{2}\pi & 0 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 0 & 0 & 2\pi & 8\pi \end{bmatrix}$? **Answer:** Yes, $\mathbf{u} = \mathbf{v} + 2\mathbf{w} + \frac{1}{2}\mathbf{x}$.

A set of vectors is called **linearly independent** if none of the vectors is a linear combination of any of the other vectors. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is called linearly independent if the only solution to the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k = \mathbf{0}$$

is $c_1 = c_2 = \dots = c_k = 0$. If another solution exists, the set of vectors is called **linearly dependent**. So, a set S of vectors is linearly dependent if at least one of the vectors in S can be written as a linear combination of the other vectors in S .

Notice that linear independence and linear dependence are defined only for collections of vectors that have the same dimension. It's incoherent to ask if $\begin{bmatrix} 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 33 & 33 & 33 \end{bmatrix}$ are linearly independent, because the addition of these vectors is not defined.

Question: Are the following sets of vectors linearly independent?

1. $\mathbf{a} = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 & 6 & 1 \end{bmatrix}$, **Answer:** Yes. But if we switched the last element in \mathbf{b} to a 2 then $\mathbf{a} = \frac{1}{2}\mathbf{b}$.
2. $\mathbf{a} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 & 5 & 0 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 10 & 10 & 0 \end{bmatrix}$, **Answer:** No, $10\mathbf{a} + 2\mathbf{b} = \mathbf{c}$.
3. $\mathbf{a} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 & 1 \end{bmatrix}$ **Answer:** Yes.
4. $\mathbf{a} = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 & 3 & 6 & 9 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} -1 & 2 & 4 & 6 \end{bmatrix}$. **Answer:** No, $\mathbf{b} = 3\mathbf{a}$.

The set of all linear combinations of a collection of vectors is called their **span**. A set B of linearly independent vectors is called the **basis** of a set V of vectors¹ if the span of B is equal to V .

Example: $\mathbf{e}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ span the set of all dimension 2 vectors. To understand why, consider any generic dimension two vector, $\mathbf{a} = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$, with $a_1, a_2 \in \mathbb{R}$. No matter the value of a_1 and a_2 , we can always write $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$. So any dimension 2 vector is a linear combination of \mathbf{e}_1 and \mathbf{e}_2 . For this reason, the set of vectors \mathbf{e}_i for each i in $\{1, 2, \dots, n\}$ (the set of **unit vectors**) is treated as the default basis for \mathbb{R}^n , $n \in \mathbb{N}$.

We can understand this visually (**Draw it**).

¹Actually, a vector space, which we won't define in these notes.

Question: Can we increase the size of the set that is being spanned by introducing linearly dependent vectors into our basis?

Answer: Nope.

We'll talk a lot more about this in 598.

6.2 Matrices

A **matrix** is a collection of real numbers arranged in a grid of m rows and n columns. In my humble opinion matrices are the absolute core of empirical social science, because datasets are simply matrices. Every empirical analysis in every published paper you will ever read is a mathematical statement about the characteristics of a matrix. Many of the assumptions of common methods are explicitly stated as conditions that a matrix of data must fulfill in order for the tool to be applicable.

We can also talk about one number in a matrix as being an **element** of the matrix. Like vectors, we use subscripts to refer to elements, but now we need two subscripts: one to identify the row and another to identify the column, and typically we use the un-bolded lowercase version of whatever bold uppercase letter we're using for the matrix. For example, in the following matrix,

$$\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

we would say that $a_{11} = 4$, $a_{12} = 3$, $a_{21} = 2$, $a_{22} = 1$. The subscript is called an **index**, so we would say that the number 2 is at “index (2,1) of \mathbf{A} ”.

Clearly matrices are closely related to vectors: in fact, you might think of vectors as a special case of matrices, since a column vector of dimension k is a $k \times 1$ matrix, while a row vector of dimension k is a $1 \times k$ matrix.

6.3 Matrix properties

Scalar multiplication for matrices is similar to our definition for vectors: for the matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $r\mathbf{A} = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}$

Example: To multiply the example matrix above by 3,

$$3\mathbf{A} = 3 \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

$$3\mathbf{A} = \begin{bmatrix} 3 \cdot 4 & 3 \cdot 3 \\ 3 \cdot 2 & 3 \cdot 1 \end{bmatrix}$$

$$3\mathbf{A} = \begin{bmatrix} 12 & 9 \\ 6 & 3 \end{bmatrix}$$

Matrix addition and **matrix subtraction** are defined exactly as you would hope: for matrices $\mathbf{A}_{m \times n}$ and $\mathbf{B}_{m \times n}$, with

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$$

their sum is simply the sum of the corresponding elements, so

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

Notice that matrix addition is only defined for matrices of the same shape.

Matrix multiplication is defined as follows: for $\mathbf{A}_{m \times n}$ and $\mathbf{B}_{n \times k}$, with

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nk} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} a_{11} \cdot b_{11} + \dots + a_{1n} \cdot b_{n1} & a_{11} \cdot b_{12} + \dots + a_{1n} \cdot b_{n2} & \dots & a_{11} \cdot b_{1k} + \dots + a_{1n} \cdot b_{nk} \\ a_{21} \cdot b_{11} + \dots + a_{2n} \cdot b_{n1} & a_{21} \cdot b_{12} + \dots + a_{2n} \cdot b_{n2} & \dots & a_{21} \cdot b_{1k} + \dots + a_{2n} \cdot b_{nk} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \cdot b_{11} + \dots + a_{mn} \cdot b_{n1} & a_{m1} \cdot b_{12} + \dots + a_{mn} \cdot b_{n2} & \dots & a_{m1} \cdot b_{1k} + \dots + a_{mn} \cdot b_{nk} \end{bmatrix}$$

How do humans remember this? Here's what I always tell myself. The top-left index is the first row by the first column. Moving to the right, the next index is the first row by the second column. And so on. The leftmost index of the second row is the second row by the first column. And so on.

Notice the convention is to write matrices using bolded capital letters.

Example: $\begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 6 \\ -1 & 9 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + 4 \cdot (-1) & 2 \cdot 6 + 4 \cdot 9 \\ 5 \cdot 3 + 3 \cdot (-1) & 5 \cdot 6 + 3 \cdot 9 \end{bmatrix} = \begin{bmatrix} 2 & 48 \\ 12 & 57 \end{bmatrix}$

Example:

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 5 \\ 4 & -3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-2) + 2 \cdot 4 + (-1) \cdot 2 & 1 \cdot 5 + 2 \cdot (-3) + (-1) \cdot 1 \\ 3 \cdot (-2) + 1 \cdot 4 + 4 \cdot 2 & 3 \cdot 5 + 1 \cdot (-3) + 4 \cdot 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 6 & 16 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 1 \cdot 2 & 1 \cdot 3 + 1 \cdot 3 \\ 2 \cdot 2 + 2 \cdot 2 & 2 \cdot 3 + 2 \cdot 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 8 & 12 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 5 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 6 + 0 \cdot 7 & 1 \cdot 5 + 0 \cdot 1 \\ 0 \cdot 6 + 1 \cdot 7 & 0 \cdot 5 + 1 \cdot 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 5 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 7 & 1 \end{bmatrix}$$

Notice two things:

1. Matrix multiplication is not necessarily commutative! That is, it is not generally (but can sometimes be) true that $\mathbf{AB} \neq \mathbf{BA}$

2. In the shapes of the matrices, $m \times n$ and $n \times k$, the inside numbers n and n need to match in order to multiply them, and the result will have the shape of the outside letters, so $m \times k$.

Noncommutative example: We saw that

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 8 & 12 \end{bmatrix}$$

But

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 3 \cdot 2 & 2 \cdot 1 + 3 \cdot 2 \\ 2 \cdot 1 + 3 \cdot 2 & 2 \cdot 1 + 3 \cdot 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$$

How do we know if we have two matrices that cannot be multiplied? We will have either too few or too many numbers in a row, when we go to take the inner product of that row with the corresponding column of the other matrix.

- $(AB)C = A(BC)$ for matrices A, B, C with $A_{m \times n}$, $B_{n \times k}$, and $C_{k \times p}$
- $(A + D)G = AG + DG$ for matrices A, D, G with $A_{m \times n}$, $D_{m \times n}$, and $G_{n \times k}$. **Question:** Verify that the shapes work out.
- $xAB = (xA)B = A(xB) = ABx$ for matrices A, B with $A_{m \times n}$, $B_{n \times k}$, and $x \in \mathbb{R}$

Question: Verify all of those properties for the following values

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, G = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, x = 3$$

As practice for our new idea of matrix multiplication, let's go back and reconsider a claim that we saw before: that for column vectors a and b , $a'b$ is their dot product. Consider

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Then

$$\mathbf{a}' = [a_1 \quad a_2 \quad \dots \quad a_n]$$

So, by our definition of matrix multiplication,

$$\mathbf{a}'\mathbf{b} = a_1b_1 + a_2b_2 + \dots + a_nb_n$$

Which is just how we defined the dot product. Similarly, you can see that $\mathbf{a}'\mathbf{a}$ captures something like the idea of squaring the column vector \mathbf{a} – it gives the sum of the square of each number in \mathbf{a} .

The $m \times m$ **identity matrix** is the matrix $\mathbf{I}_{m \times m}$ such that $\mathbf{I}\mathbf{A} = \mathbf{A}\mathbf{I} = \mathbf{A} \quad \forall \mathbf{A}_{m \times m}$

The identity matrix has the form:

$$\mathbf{I}_{m \times m} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Example: Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then

$$\mathbf{I}_{2 \times 2}\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\mathbf{I}_{2 \times 2} \mathbf{A} = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 3 & 1 \cdot 2 + 0 \cdot 4 \\ 0 \cdot 1 + 1 \cdot 3 & 0 \cdot 2 + 1 \cdot 4 \end{bmatrix}$$

$$\mathbf{I}_{2 \times 2} \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Notice that identity matrices are always square. **Question:** Why? **Answer:** Because if you multiply a nonsquare matrix by a nonsquare matrix you necessarily get a different matrix because the shape will change.

If an operation of interest is defined on two matrices, we call those matrices **conformable**.

A matrix \mathbf{A} is called **idempotent** if $\mathbf{A}\mathbf{A} = \mathbf{A}$.

Example: Let $\mathbf{A} = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}$. Then

$$\mathbf{A}\mathbf{A} = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}$$

$$\mathbf{A}^2 = \begin{bmatrix} 4 \cdot 4 + (-1) \cdot 12 & 4 \cdot (-1) + (-1) \cdot (-3) \\ 12 \cdot 4 + (-3) \cdot 12 & 12 \cdot (-1) + (-3) \cdot (-3) \end{bmatrix}$$

$$\mathbf{A}^2 = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}$$

$$\mathbf{A}^2 = \mathbf{A}$$

The **matrix transpose** of the matrix \mathbf{A} , denoted \mathbf{A}^T or \mathbf{A}' , is an operation that switches the rows and columns of a matrix \mathbf{A} .

Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \implies \mathbf{A}' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Properties of matrix transposes, with conformable matrices \mathbf{A}, \mathbf{B} and $c \in \mathbb{R}$:

- $(\mathbf{A}')' = \mathbf{A}$
- $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$

- $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$
- $(c\mathbf{A})' = c\mathbf{A}'$

The **rank** of a matrix is the number of linearly independent rows or columns (these are the same).

Example: The following matrix has rank 3:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The following matrix has only rank 2, because its third row is a linear combination of the first row and the second row:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Practice questions:

1. Is \mathbf{I} idempotent? **Answer:** Sure is.
2. Compute $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ or explain why you cannot
3. Compute $\begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 2 \\ 2 & 0 \end{bmatrix}$ or explain why you cannot
4. Compute $\begin{bmatrix} 1 & 3 \\ 4 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 0 \end{bmatrix}$ or explain why you cannot
5. Compute $\begin{bmatrix} 1 & 3 \\ 4 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ or explain why you cannot
6. Compute $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}'$
7. Compute $3 \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix}'$

8. What is the rank of each of the following matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ Answer: } 2$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ Answer: } 1$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \text{ Answer: } 1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 8 \end{bmatrix} \text{ Answer: } 2$$

6.4 Matrix inversion

We've developed three of the four standard binary arithmetical operators for matrices: we can add, subtract, and multiply. So what about division? We won't develop any general idea of matrix division, but we will develop the special case of multiplying by a reciprocal: namely, what do we have to multiply a matrix by to get the matrix equivalent of the number one? The equivalent of a reciprocal for a matrix \mathbf{A} is called its inverse.

It turns out that some of the most important tools in political science – including linear regression – involve taking the inverse of a matrix of data. The inverse of a matrix \mathbf{A} , denoted \mathbf{A}^{-1} , is the matrix such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. In 598 we will talk a *lot* about how to calculate matrix inverses.

Example: Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. I claim that the inverse of this matrix is

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}. \text{ Let's verify that assertion:}$$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2} - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example: Sadly (?), not all matrices have inverses! First, only square matrices can have inverses. But also not all square matrices have inverses. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 5 \end{bmatrix}$$

If \mathbf{A} has an inverse \mathbf{A}^{-1} , then it has to satisfy the definition of an inverse, meaning:

$$\mathbf{A}\mathbf{A}^{-1} = I_{2 \times 2}$$

Then

$$\begin{bmatrix} 0 & 0 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ a + 5c & b + 5d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So for any values of a, b, c, d , we will require $0 = 1$. Thus there can be no matrix that satisfies the requirements for \mathbf{A}^{-1} in this example.

The reason that this did not work out is that the top row in \mathbf{A} contains only zeroes. But that is not the only problem that can stop a matrix from having an inverse, it is only one example of a problem.

So how can we calculate this peculiar beast? And when does it exist? To answer these questions, we'll have to first develop a few more ideas.

6.4.1 Determinants and adjoints

The definition for the **determinant** of a matrix depends on its size (or else requires many more definitions than we have time to discuss here). In math camp, I just want you to see the determinant of a 2 by 2 matrix, so that when you see the determinants of larger matrices in (especially) 598 and 699 you will have some basis for comparison. That definition is as follows.

The **determinant** of the real-valued 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is denoted $\det \mathbf{A}$ or $|\mathbf{A}|$ or (more fully) $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and is given by

$$\det \mathbf{A} = ad - bc$$

It turns out that the determinant of a matrix holds the secret to whether or not it is invertible. Let's practice calculating them so that we're ready when we see the role they play.

Example: Let

$$\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$$

then

$$\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = 5 \cdot 3 - 4 \cdot 2$$

$$\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = 7$$

Example: Let's try another

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

then

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \cdot 1 - 0 \cdot 0$$

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Finally, before we find out how to calculate matrix inverses, we need one more idea: the matrix adjoint. The adjoint also is defined for a matrix with any number of entries, but we will stick to thinking about adjoints of 2 by 2 matrices for today.

The **adjoint** of the real-valued 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is given by

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example: Let

$$\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$$

then the adjoint of \mathbf{A} , $\text{adj } \mathbf{A}$, is given by

$$\text{adj } \mathbf{A} = \begin{bmatrix} 3 & -4 \\ -2 & 5 \end{bmatrix}$$

Example: Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

then

$$\text{adj } \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

6.4.2 Equation for inverting a matrix

At long last, we are ready to see the equation for inverting a square matrix \mathbf{A} . Hurray!

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \text{adj } \mathbf{A}$$

This shows the crucial test for whether or not a matrix has an inverse. If its determinant equals zero, then the equation for its inverse is not defined, because it involves division by zero. If its determinant is nonzero, then we will always be able to calculate an inverse using this formula.

Let's see some examples.

Example: Find the inverse of

$$\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$$

We have already shown that

$$\text{adj } \mathbf{A} = \begin{bmatrix} 3 & -4 \\ -2 & 5 \end{bmatrix}$$

and

$$\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = 7$$

So we can find the inverse by using our inverse formula,

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \text{adj } \mathbf{A}$$

$$\mathbf{A}^{-1} = \frac{1}{7} \text{adj } \begin{bmatrix} 3 & -4 \\ -2 & 5 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{3}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{5}{7} \end{bmatrix}$$

So, did it work? Let's check if this was a successful inverse using the definition of a matrix inverse.

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{3}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{5}{7} \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 5 \cdot \frac{3}{7} - 4 \cdot \frac{2}{7} & 5 \cdot -\frac{4}{7} + 4 \cdot \frac{5}{7} \\ 2 \cdot \frac{3}{7} - 3 \cdot \frac{2}{7} & 2 \cdot -\frac{4}{7} + 3 \cdot \frac{5}{7} \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} \frac{15}{7} - \frac{8}{7} & -\frac{20}{7} + \frac{20}{7} \\ \frac{6}{7} - \frac{6}{7} & -\frac{8}{7} + \frac{15}{7} \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} \frac{7}{7} & 0 \\ 0 & \frac{7}{7} \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^{-1} = I$$

Huzzah! Now, we also said one other thing should work. We should also be able to *left*-multiply \mathbf{A} by \mathbf{A}^{-1} , and find that $\mathbf{A}^{-1}\mathbf{A} = I$. **Question:** Verify this on your own.

Note: Notice that, the way we defined everything, whenever \mathbf{A}^{-1} exists, then $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

So in this example the inverse of $\begin{bmatrix} \frac{3}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{5}{7} \end{bmatrix}$ is $\begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$.

Example: Let's find the inverse of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Start by finding the determinant:

$$\det \mathbf{A} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$\det \mathbf{A} = 2 \cdot 2 - 1 \cdot 1$$

$$\det \mathbf{A} = 3$$

Next find the adjoint

$$\text{adj } \mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Now we can apply the inverse formula

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \text{adj } \mathbf{A}$$

$$\mathbf{A}^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Now, if you're feeling plucky, you can verify that $\mathbf{A}\mathbf{A}^{-1} = I$:

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 2\frac{2}{3} - \frac{1}{3} & -2\frac{1}{3} + \frac{2}{3} \\ \frac{2}{3} - 2\frac{1}{3} & -\frac{1}{3} + 2\frac{2}{3} \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

6.4.3 Eigenvalues and eigenvectors

Let \mathbf{A} be a square matrix. A number λ is an **eigenvalue** of the matrix \mathbf{A} if the system of linear equations

$$\mathbf{A}\mathbf{X} = \lambda\mathbf{X}$$

has nonzero solutions \mathbf{X} , called **eigenvectors**. Notice that we can rewrite this equation as follows:

$$\lambda\mathbf{X} - \mathbf{A}\mathbf{X} = 0$$

$$\lambda I\mathbf{X} - \mathbf{A}\mathbf{X} = 0$$

Right-factoring out the \mathbf{X} (when you do this step, make sure you don't accidentally switch the order of the matrices — the order in which you multiply matrices matters!)

$$(\lambda I - \mathbf{A})\mathbf{X} = 0$$

The reason I show this to you now is that this equation has a tight connection to the existence of a matrix inverse. Specifically, $\lambda I - \mathbf{A}$ represents the matrix determinant, and the

equation above has a nonzero solution for \mathbf{X} if and only if the determinant is zero. A matrix has an inverse if and only if it does not have zero as an eigenvalue.

Example: We will show that $\lambda = -3$ is an eigenvalue of the matrix²

$$\mathbf{A} = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$$

We verify that $\lambda = -3$ satisfies the condition $(\lambda I - \mathbf{A})\mathbf{X} = \mathbf{0}$. So,

$$\left(-3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -8 & -8 & -16 \\ -4 & -4 & -8 \\ 4 & 4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So this is the specific claim we're making. It turns out that any \mathbf{X} is a solution that satisfies

$$\mathbf{X} = a_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

with $a_1, a_2 \in \mathbb{R}, a_1, a_2 \neq 0$.

6.5 Elementary row operations and matrix factorisations

Now that we have a good understanding of matrices for their own sake, let's imbue them with a very common interpretation: thinking about matrices as a container that represents

²This example is taken from *Linear algebra with applications* (1986) by W. Keith Nicholson.

a solution to a system of linear equations.

Let's consider the following system of equations:³

$$\text{Equation 1: } 3x + 4y + z = 1$$

$$\text{Equation 2: } 2x + 3y = 0$$

$$\text{Equation 3: } 4x + 3y - z = -2$$

Let's solve it in the way we learned in school: using substitution. By Equation 2, we have

$$3y = -2x$$

$$y = -\frac{2}{3}x$$

Inserting this into Equation 1,

$$3x + 4\left(-\frac{2}{3}\right)x + z = 1$$

$$3x - \frac{8}{3}x + z = 1$$

$$\frac{1}{3}x + z = 1$$

$$\frac{1}{3}x = 1 - z$$

$$x = 3 - 3z$$

³This example is heavily adapted from an example in *Linear algebra with applications* (1986) by W. Keith Nicholson.

Now inserting these into, say, Equation 3:

$$4x + 3y - z = -2$$

$$4(3 - 3z) + 3\left(-\frac{2}{3}(3 - 3z)\right) - z = -2$$

$$12 - 12z - 6 + 6z - z = -2$$

$$-7z = -8$$

$$z = \frac{8}{7}$$

So that we have

$$x = 3 - 3\frac{8}{7}$$

$$x = \frac{21}{7} - \frac{24}{7}$$

$$x = -\frac{3}{7}$$

And

$$y = -\frac{2}{3}\left(-\frac{3}{7}\right)$$

$$y = \frac{2}{7}$$

The extreme horror of solving this system by substitution is our main motivation for wanting to use matrices to solve systems of linear equations. Imagine if we had 55 variables and 100 equations, it could take us all day to solve a big system by substitution. To that end, we want to develop a way to do this kind of work much faster using matrices. First, let's write that system of equations as a matrix, with the coefficients of each variable (*in order!*) inside the bulk of the matrix, and the solution to that system of equations demarcated over on the right-hand side.

$$\left(\begin{array}{ccc|c} 3 & 4 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \end{array} \right)$$

What can we do to this matrix without losing the meat of what the equations are saying? I'm going to make a claim, and then we're going to see what that claim is true.

The **elementary row operations** are as follows:

- Switch two rows
- Multiply a row by a nonzero scalar
- Add a multiple of one row to another

Let's see why the elementary row operations are OK to apply to a system of linear equations. We said they are tools for simplifying a system of equations without changing the result. **Question:** Try to change the result by adding a multiple of one of these equations to the other, multiplying both sides of one of the equations by a nonzero scalar, or multiplying both sides of an equation by 0. It should be clear that switching the order of the equations certainly does not change the answer we will get.

Our first job is to get a 1 in the top-left index. A very natural first step is to subtract row 2 from row 1:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \end{array} \right)$$

Now we can use this newfound 1 to reduce the next two rows to zero, first by subtracting 2 times the first row from the second row:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 4 & 3 & -1 & -2 \end{array} \right)$$

And next by subtracting 4 times row 1 from row 3:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -5 & -6 \end{array}\right)$$

Now these other rows can be used to get some zeroes in the digits behind the leading digits. First let's subtract row 2 from row 1:

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -5 & -6 \end{array}\right)$$

Adding row 2 to row 3 will produce a similar effect:

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -7 & -8 \end{array}\right)$$

We want to get this into the simplest form possible, so the next step should be to adjust row 3 so that its only remaining coefficient is 1, by multiplying it by the scalar $-\frac{1}{7}$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & \frac{8}{7} \end{array}\right)$$

Adding two times row 3 to row 2 produces the following result:

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & 0 & \frac{2}{7} \\ 0 & 0 & 1 & \frac{8}{7} \end{array}\right)$$

And finally, subtracting 3 times row 3 to row 1 produces the reduced row echelon form augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{3}{7} \\ 0 & 1 & 0 & \frac{2}{7} \\ 0 & 0 & 1 & \frac{8}{7} \end{array}\right)$$

If we split this apart into equations again, we get the three equations

$$x = -\frac{3}{7}$$

$$y = \frac{2}{7}$$

$$z = \frac{8}{7}$$

So we can see that it matches the direct result we got by substitution. This method will always work for any system of linear equations that has a solution. The following definition explains what we want the matrix to look like so that we can read the solution to the system of linear equations off:

A matrix is in **row-echelon form** if all of the following are true:

- The leftmost nonzero entry in each row (called the **leading entry**) is a 1,
- Every leading entry is to the right of the leading entry above it,
- Rows that only contain zeroes are at the bottom of the matrix.

A matrix is in **reduced row-echelon form** if

- It is in row-echelon form,
- Each leading 1 is the only nonzero entry in its column

Example: Solve the following system of equations by setting up an augmented matrix and row reducing it:

$$x_1 + 2x_2 = 3$$

$$x_1 + 3x_2 = 5$$

The augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 3 & 5 \end{array}\right)$$

First, I want to get rid of that 1 in the bottom left. That looks easy enough to do: just subtract row 1 from row 2.

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 3 & 5 \end{array}\right)$$

$$\xrightarrow{R_2 - R_1 \rightarrow R_2} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 1 - 1 & 3 - 2 & 5 - 3 \end{array}\right)$$

$$\xrightarrow{R_2 - R_1 \rightarrow R_2} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 2 \end{array}\right)$$

Next, to simplify the matrix, we want to get rid of that 2 over the 1. To do that, we can simply take 2 times row 2 away from row 1:

$$\xrightarrow{R_1 - 2R_2 \rightarrow R_1} \left(\begin{array}{cc|c} 1 & 2 - 2 \cdot 1 & 3 - 2 \cdot 2 \\ 0 & 1 & 2 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right)$$

Now we can read off the answers:

$$x_1 = -1$$

$$x_2 = 2$$

Example: Solve the following system of equations by setting up an augmented matrix and row reducing it:

$$2x_1 + 3x_2 = 5$$

$$4x_1 - 2x_2 = 3$$

The augmented matrix is

$$\left(\begin{array}{cc|c} 2 & 3 & 5 \\ 4 & -2 & 3 \end{array} \right)$$

$$\xrightarrow{R_2 - 2R_1 \rightarrow R_2} \left(\begin{array}{cc|c} 2 & 3 & 5 \\ 4 - 2 \cdot 2 & -2 - 3 \cdot 2 & 3 - 2 \cdot 5 \end{array} \right)$$

$$\left(\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & -8 & -7 \end{array} \right)$$

$$\xrightarrow{-\frac{1}{8}R_2 \rightarrow R_2} \left(\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & 1 & \frac{7}{8} \end{array} \right)$$

$$\xrightarrow{R_1 - 3R_2 \rightarrow R_1} \left(\begin{array}{cc|c} 2 & 0 & 5 - \frac{21}{8} \\ 0 & 1 & \frac{7}{8} \end{array} \right)$$

$$\left(\begin{array}{cc|c} 2 & 0 & \frac{19}{8} \\ 0 & 1 & \frac{7}{8} \end{array} \right)$$

$$\xrightarrow{\frac{1}{2}R_1 \rightarrow R_1} \left(\begin{array}{cc|c} 1 & 0 & \frac{19}{16} \\ 0 & 1 & \frac{7}{8} \end{array} \right)$$

So $x_1 = \frac{19}{16}$ and $x_2 = \frac{7}{8}$. You could confirm this by substitution if you were so motivated.

Question: Use elementary row operations to bring the following matrices to reduced row echelon form, then check your work with Wolfram|Alpha using syntax like the example below.

$$\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 5 & 3 & 2 \end{array} \right)$$

row reduce $\{\{1,2,5\},\{5,3,2\}\}$

$$\left(\begin{array}{ccc|c} 1 & 0 & 4 & 5 \\ 9 & 9 & 9 & 1 \\ 1 & 1 & 1 & 9 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 7 & 3 \end{array}\right)$$

Question: What is wrong with the following system of linear equations?

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 1 \end{array}\right)$$

Answer: It requires that $0x_1 + 0x_2 = 1$, so $0 = 1$. This tells us that this system of linear equations has no solution.

7 Logic and common types of proof

7.1 Logic

In the methods sequence, and on occasion in published research, you will encounter notation from formal logic. It's good to have seen the basics once:

- “ $A \implies B$ ”, read as “ A implies B ”, means that B is true whenever A is true. So, “if A , then B ”
- If $A \implies B$ and $B \implies A$, then we write $A \iff B$, read as “ A if and only if B ” and also written “ A iff B ”. This means that A and B always have the same truth value: if A is true then B is true, if A is false then B is false, if B is true then A is true, and if B is false then A is false. If you haven't thought about this before, it's worth setting aside a moment to think about why “if and only if” is an appropriate phrase for that.
- “Not A ” or “the negation of A ” is denoted $\neg A$. Whatever the truth value of A is, $\neg A$ has the opposite truth value.
- We said that $A \implies B$ means that whenever A is true, B will also have to be true. So what if we observe that B is not true? This is called the contrapositive: if A forces B to be true, and B is not true, then A cannot be true either.
- “And” is denoted \wedge
- “Or” is denoted \vee
- We can also create composite statements, where one proposition affects the relationship between other propositions: $A \implies (B \implies C)$: the proposition A implies that the proposition B implies C .

Common terms in proofs and derivations:

- Axiom or assumption: a statement used before or at the start of a proof that is taken to be true.
- Theorem: a proven proposition, or a statement of interest to be proven.
- Lemma: an intermediate claim, which can be already proven, simply assumed or proven in the midst of another proof, which acts as a step in the proof of a theorem.

- **Corollary:** a claim that follows from another claim, and does not require an independent proof.

7.2 Three ubiquitous types of proof

- **Direct:** use deduction to string together series of true statements, starting with the assumptions and ending with the conclusion.
- **Contradiction:** start by assuming the statement you’re trying to prove is actually false, then show that this implies a contradiction.
- **Counterexample:** suppose we make a general claim about a big family of objects. If we can find a single object in that family of objects that the statement is not true of, then the general claim cannot be true.

Let’s talk a bit about how to approach each of these types of demonstration.

7.2.1 How to approach “Show that this is true”

Fairly often in classes, most likely in the stats class POLSCI 599, you will encounter questions like “show that [some statement] is true” or “show that [some equation] is equal to [some other equation]”. What do we do when we see these sorts of questions?

I’m going to modify some real examples from past offerings of POLSCI 599, swapping out exactly *what* is being established, but keeping the form of the question. We will practice a technique for tackling these sorts of questions.

Imagine that you were asked “show that $2^2 = 4$ ”. I recommend the following simple technique:

Step 1: Ask yourself what the starting point is. Here, we are supposed to start with 2^2 .

Step 2: Ask yourself what the goal is. Here, the goal is to do something to 2^2 and arrive at the number 4.

Step 3: Now physically write down the starting point and the goal, demarcated by a line, as follows:

LHS	RHS
2^2	4

Once you have your starting point written down on the left, you know exactly what you have to fiddle around with in order to arrive at the desired goal. Here, I want to take 2^2 and

rearrange it according to whatever relevant rules I know until I get the number 4.

In this example, my first idea would be to recall the definition of exponentiation in terms of multiplication. That is, I can rewrite the value on the next line as follows:

LHS	RHS
2^2	4
$2 \cdot 2$	

And now all that remains is to do the multiplication:

LHS	RHS
2^2	4
$2 \cdot 2$	
4	

When statement you're manipulating matches the goal, you're done.

Let's try a more realistic example. Say that you have the sets $S = \{1, 2, 3\}$ and $T = \{3, 4, 5\}$ and you are asked the following question: "Show that $(S \cup T) \setminus (S \cap T) \cup \emptyset = \{1, 2, 4, 5\}$." Let's start by just writing the question out in the format for derivations, with the starting place on the left-hand side and the goal on the right-hand side:

LHS	RHS
$(S \cup T) \setminus (S \cap T) \cup \emptyset$	$\{1, 2, 4, 5\}$

Now I'm going to substitute in the values as given in the question:

LHS	RHS
$(S \cup T) \setminus (S \cap T) \cup \emptyset$	$\{1, 2, 4, 5\}$
$(\{1, 2, 3\} \cup \{3, 4, 5\}) \setminus (\{1, 2, 3\} \cap \{3, 4, 5\}) \cup \emptyset$	

Now, suppose we're stuck. What do we do next? There are several symbols that we have to make sure we understand before we can proceed. In this case, we want to make sure we remember the meanings of \cup , \cap , \setminus , and \emptyset . Other good things to do in a situation like this

are to try to rephrase the expression using other (maybe simpler) versions of it that you've seen elsewhere; for example, maybe $\{1, 2, 3\}$ is a set you have seen an expression for in some slide somewhere, and it turns out you can do something simplifying to it. In this situation, all we need to do is make sure we remember our operations and apply them, as follows:

LHS	RHS
$(S \cup T) (S \cap T) \cup \emptyset$ $(\{1, 2, 3\} \cup \{3, 4, 5\}) \setminus (\{1, 2, 3\} \cap \{3, 4, 5\}) \cup \emptyset$ $\{1, 2, 3, 4, 5\} \setminus (\{1, 2, 3\} \cap \{3, 4, 5\}) \cup \emptyset$ $\{1, 2, 3, 4, 5\} \setminus \{3\} \cup \emptyset$ $\{1, 2, 4, 5\} \cup \emptyset$ $\{1, 2, 4, 5\}$	$\{1, 2, 4, 5\}$

Now that we have the same thing on the left and the right, we have shown the result.

7.2.2 How to approach “Show that this is false”

Often this type of statement will be amenable to a direct approach. But there are two special tricks that work here which specifically work for showing that a statement is false.

Example of a counterexample: Suppose you encounter the question “prove that not all natural numbers are odd.” You could proceed as follows. First, notice that the negation of the sentence is “all natural numbers are odd”. It must be true either that all numbers are odd, or that not all numbers are odd. So I want to disprove the statement “all natural numbers are odd”. To disprove this statement it suffices to give a counterexample. If I can name a natural number that is not odd, then it cannot be true that all natural numbers are odd. So I will pick the number 4. Since 4 is not odd, it is not true that all natural numbers are odd, hence we have proven that not all natural numbers are odd.

Notice, however, that examples are different from counterexamples!

Example that doesn't work: Suppose you encounter the question “prove that all integers are positive”, and you say “indeed, consider 5. 5 is a positive integer. Therefore all integers are positive”. This is not just wrong, it is actually not even a response to the question you were asked. The fact that you can come up with examples of positive integers says nothing about the truth of statement that *all* integers are positive! It just means that there's at least one integer that's positive. Indeed, the statement can be *disproven* with the counterexample of, say, -2.

It is extremely common for people to try to prove general statements in the methods sequence with answers that start out “Consider the example ...”, and then talk only about one example for which the statement happens to be true. If you can find an example for which a general statement is wrong, then that statement must be wrong. But if you find an example for which a general statement is true, all you know is that the statement is not wrong *for every single possible example*; it still could be wrong about arbitrarily many other examples.

Contradiction is arguably easier to wrap your head around. The idea is as follows. Say that you encounter a question like “Show that [some statement] is false.” The trick in proof by contradiction is to first assume that the statement is actually true, and then follow a series of steps each of which is individually true. If eventually you hit a contradiction — a falsehood — then something in your argument must have been false. Assuming that every step along the way really was true — that is, that you didn’t make any mistakes — then the only thing that can be wrong is the assumption that the statement is true, so the statement must have been false instead!

Here’s a simple example of a proof by contradiction using material that we’re familiar with. I might ask you the following: Prove that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is not the inverse of

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

You could proceed as follows. I know that either a given matrix is the inverse of a certain matrix, or it is not. There’s no other possibility: it is or it isn’t. So what happens if we suppose that the statement is true? Let’s suppose that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is indeed the inverse of

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then, according to our definition of a matrix inverse, it must be true that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

but if you conduct the matrix multiplication, you will see that the left-hand side is not equal to the right-hand side:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We arrived at an abject falsehood, so something that we did along the way must have been incorrect, otherwise how could we have arrived at a falsehood? To understand where we went wrong, let's think through everything we did and see what happened. We are confident that we did our matrix multiplication right in the last step. We also know for sure that we got the definition of a matrix inverse right, so we set up the equation correctly and then we reduced it correctly until it was obviously wrong. The only possible falsehood was the first step: we must have been wrong in assuming that the given matrices were inverses of each other, because they didn't satisfy the one requirement of an inverse!

This was a proof by contradiction: we assumed the negation of the proposition, reasoned through it until we hit something that was obviously wrong (the eponymous "contradiction"), and realised that the negation of the proposition must be false, so the only possible conclusion is that the original proposition is true.