Equilibrium cycles in iterative voting games with simultaneous updating

Samuel Baltz

May 2019

Abstract

I propose a notion of equilibrium cycles that bridges a disparity between the structure of analytical game theory and iterative computer models. The presence of endless cycles in the end-states of many computational formal models makes it prohibitively difficult to compare them to game theory models, but a detailed study of the possible cycles in best-response equilibria reveals strongly simplifying patterns. Using iterative voting games as a test case, I show that my notion of iterative equilibrium can be used as a meaningful stopping condition when modeling political processes in the computer. I state and prove a series of theorems that constrain the possible patterns in iterative voting games, and I check and illustrate these theorems using eight novel computer models.

1 Introduction

In an election, a large group of people receive updates about which alternatives are competitive, and these updates inform their vote choice. Updates may be polls, new policy announcements, conversations with friends, or nightly news casts. For simplicity we might think of these updates as being offered to every person at once, in which case electors have a sequence of simultaneous opportunities to re-evaluiate their vote choices. After receiving sufficiently many updates, we might hope that every elector will have enough information to make a firm decision, and will settle down on a vote choice; the situation in which electors will not change their mind regardless of how many more updates they are shown is called an equilibrium. The typical conceit of formal models in this setting is that, by the time of the election, everyone should have had the opportunity to receive sufficiently many updates, and so the equilibrium is the natural choice for an election result.

This picture has two deep methodological problems. First, current tools in the formal modeling of political phenomena are not designed to model large sequences of updates over time. In game theory, which is the major approach to modeling elections, time is typically abstracted away, and it is rare to explicitly model the regular information updates that characterise real election periods. Rather, it is assumed that the electorate will arrive at some stable Nash Equilibrium state, whether through the conventional conceit that the players meet "in the bar the night before" to work out what the best strategy for every player is, or for the same reason that populations of animals arrive at equilibrium states over time (Binmore, 1990: p. 61). The second deep methodological problem, however, is that even if we do explicitly model a notion of time, there is no reason to believe that the electorate should always arrive at a stable state in which no elector wishes to change their opinions. Indeed, elections are famously characterized above all else by vast numbers of possible equilibria, so in iterative election games with simultaneous updating it seems implausible that the electorate will always arrive at one rigid equilibrium.

The possibility of equilibrium cycles represents a sharp methodological discrepancy between conventional game theory models and any model that seeks to capture the reality that elections feature frquent updates. As political scientists have begun to design computational election models in which agents iteratively select best response strategies through a sequence of discrete time steps, a natural starting point is to translate an existing game theory model into the computer, effectively creating a computational example generator of the game theory model. The researcher can then combine multiple game theory models to produce a computational formal model that may not have been analytically solvableu using pencil and paper. Unfortunately, where the game theory model produces one single equilibrium state, the iterative model may arrive at a never-ending cycle of equilibria. Even when a computational model and a game theoretic model appear to make exactly the same assumptions, and even when the players use strict best response logic, the iterative version can produce cycles of best response equilibria in situations where the pencil-and-paper model yields one unique equilibrium.

How complicated are these equilibrium cycles, and do they bear any relationship to the equilibria in the static time game? To answer these questions, I proceed in three steps. First, I define what I call an iterative equilibrium, which I argue is a state of a computational election model at which its results can be usefully interpreted. This definition is general enough that it can be applied to other simulations of political processes, not only to election models. Second, I use this definition to prove regularities in the behaviour of individuals in the voting game based on patterns in the aggregate results. Third, I check and illustrate my theorems using more than 11,000,000 runs of eight novel computational formal models.

Although my equilibrium notion is phrased in quite general terms, in that it does not include any ideas which are specific to the study of elections, in this paper I focus only on its substantive application to democratic elections with strategic voters who pick best response strategies based on their probabilities of being pivotal. This paper has two substantive goals. The primary substantive application is to enable elections researchers to develop computational formal models of elections which relax some of the assumptions that are necessary for solvability in pencil-and-paper formal treatments of voting games, for example to be able to model more realistically heterogeneous electorates. My secondary substantive motivation is that iterative voting games can be directly used to model some substantive situations. In particular, the sorts of complex dynamics that are ubiquitous in iterative games provide a new angle for studying the cyclic and chaotic results of social choice theory, and I also explore the connection between iterative voting cycles and the chaotic cycles that have been very thoroughly studied in static or repeated election games (McKelvey, 1976, Schofield, 1978). A close analogue, to which I compare my results, is found in those pencil-and-paper formal theoretic models which apply dynamical systems ideas to voting games (Acemoglu et al., 2009, Demichelis and Dhillon, 2010, Fey, 1997, Golder, 2003, Konishi and

Ray, 2003, Linzer and Honaker, 2003, Montoro-Pons and Puchades-Navarro, 2001, Sieg and Schulz, 1995).

Elections are a promising area in which to develop computational modeling methods, because many real electorates include tens of millions of diverse actors, and representing such a large complex system using canonical pencil-and-paper formal modeling techniques is intractable without introducing quite restrictive assumptions. Under specific circumstances and with sufficiently many simplifications, an impressive literature has modeled strategic voting using pivotal voting logic in the context of static time game theory to identify the mutual best response of the electors[1] in an election. The application of Nash Equilibria to elections grew out of the first attempts to model choices over parties and candidates (Downs, 1957, Riker and Ordeshook, 1968), and became explicitly calculable when authors like Palfrey and Rosenthal (1985) introduced the idea of modeling the probabilities of ties between candidates using a multinomial distribution. These multinomial tie probabilities are difficult to calculate and work with, and a series of models have subsequently offered usable alternatives. Even so, two refrains in the history of game theory models of elections are the practical difficulty of calculating these probabilities, and the strong assumptions that need to be made in order to retain pencil-and-paper solvability.

A few recent papers suggest the glimmerings of a new approach: a computational study of elections. In the 1990s, a series of papers pioneered the use of computers to understand the dynamics of party competition (Kollman et al., 1992, 1997, 1998). Although this topic of study has remained small, computational election models have been published with increasing frequency throughout the last two decades as large-scale computing power has become more available (Kim et al., 2010, Quinn and Martin, 2002, Sadiraj et al., 2004, 2006). In the last few years, several groups of political scientists have either applied computational modeling methods to party competition and voting games (Baltz et al., 2018, Bendor et al., 2011, Laver and Sergenti, 2012) or begun to study the methodology of how to recover and interpret the output of large-scale computational models of political processes (Bendor et al., 2011, Siegel, 2018). This paper builds on that small literature by describing an iterative procedure for conducting voting games in the computer, defining a method for understanding their end-states, and then using that method to prove a series of claims about the iterative system.

Because of the immense proliferation of computer simulation methods across academic disciplines, it might seem that the equilibrium state of any computer model should already be well-known and uniformly defined. Any computational model of a physical or social process must have a stopping condition, so that there is some moment when we can instruct the model to stop running, and extract meaning from that run.[2] However, there is no single best

[1] Throughout this paper I use "electors" where it might seem more natural to use "voters". I use the word "electors" deliberately, because "voters" seems to only include those people who actually cast a vote. It is crucial that when I say "electors", I mean either "voters" or "abstainers".

[2] Throughout this paper I use the words "run" and "iteration" with very specific meanings; it is worth defining them now. An iterative model consists of one set of operations which are performed sequentially in a progression of discrete iterations until some stopping condition is met. A run of a model is a collection of sequential iterations which share the same starting values. So, a run consists of a set of initial values together with a series of iterations which proceed from those starting values up until some end point. There are multiple iterations per run.

method for stopping a computer simulation. Books and articles about computer simulation methods which cover more than one field of study typically leave open the question of how to determine when a simulation should stop and deciding how to summarize all of the information that has been generated during a run (Bratley et al., 1987: ch. 3.8.1; Gilbert, 2008: p. 20; Railsback and Grimm, 2012: ch. 14.2; Weisbuch, 1991: p. 18; Zeigler, 1976: ch. 5). Instead, practitioners argue that the specific stopping condition should be motivated by the substance of the problem. Stopping conditions can therefore vary dramatically across fields of study.

This variation can be extreme, as a few examples will illustrate. In the widely used stellar astrophysics simulation suite Modules for Experiments in Stellar Astrophysics, users set starting parameters and stopping parameters to simulate stellar evolution. These stopping parameters are physical details of the star, so the stopping conditions are typically motivated by the substantive question of what phase of stellar life the user wants to examine (Paxton et al., 2010). This is in stark contrast to many cellular automata like Conway's Game of Life, where there is simply no stopping condition at all. A cellular automaton will either eventually stop changing or it will change in the same way indefinitely, with every iteration of the model taken to be interesting in its own right (Adami, 1998, Wolfram, 2002). A much greater variety exists among, for example, epidemiological models. Here, the model may stop after a certain amount of time has passed in the simulation, or the model might run until either every individual or no individual is infected. If the model runs until the number of people infected is neither increasing nor decreasing, then the problem faced by the researcher in defining an appropriate stopping condition is extremely similar to the problem that motivates this paper.

To appreciate why the general literature on computer modeling has no simple answer to this problem, consider that the question of how to determine when a run of a model is "finished" is really a fundamental question about how to derive meaning from models. If the information contained in the model in iteration t is different from the information contained in some other iteration t', then if I stop the model in iteration t I may get a different answer to my question than I would have if I had stopped the model in t'. For example, a model of an election might declare in some iteration t that candidate ξ_1 should expect to receive the highest vote total, and then declare in iteration t' that a different candidate ξ_2 should expect to receive the highest vote total.[3] The arbitrary choice of which iteration should be the last iteration can fully determine which candidate wins the election.

Very little general work has been done on defining stopping conditions in computational problems of interest to political scientists. The explicit methodological work that has been done on this question largely tries to retain the desirable properties of comparative static equilibria (Siegel, 2018). While the search for static equilibrium-like results in iterative computational models has many extremely desirable properties, like retaining many of the same intuitions that political scientists have developed for interpreting game theoretic models, a notion of dynamic equilibria is also needed in order to capture the central goal of many computational methods: to accurately represent dynamical systems (Gilbert, 2008, Railsback and Grimm, 2012, Zeigler, 1976).

The need for a direct study of the equilibria, and the regularities in strategies at equi-

[3]We will see many situations similar to this throughout this paper; one such example is Example 4.

librium, is motivated by the observation that new problems arise when attempting to state one result of an iterative voting game, and these problems do not exist in the case of static time voting games. This issue has already appeared in small pockets of the political science literature. The most obvious is the finding by Laver and Sergenti (2012) that certain static equilibrium results are not sinks of an analogous dynamical system, and that computational replications of certain game theoretic models in the party systems literature suffer from the instability problem that arbitrarily small perturbations away from the static equilibrium values produce dramatically different results than the static game theory model predicts. Regarding discrepancies they find between the long-run stable states of computer replications of game theoretic models and the game theory models themselves, they write "while situations in which parties have very different valences are modeled as equilibriums at isolated time points by static models, these situations are almost *impossible to sustain as dynamic equilibriums in an evolutionary setting*"[4] (Laver and Sergenti, 2012: p. 263). A very similar finding was made by Sadiraj et al. (2006), whose computer model of electoral competition produces ubiquitous cycling dynamics. This should certainly not be a surprise, since it is well known that discretizing a collection of differential equations can radically change the equilibria of the system, even when the time steps are very small (Dekel and Scotchmer, 1992). Indeed, the problem here is much worse: that finding holds for an otherwise identical system of differential and difference equations, which surely are dramatically more similar to one another than a pencil-and-paper game theoretic derivation and its computational analog. The notion of time in iterative models together with their reliance on finite precision arithmetic will inevitably cause entire families of equilibria to appear that cannot arise in models with static time and continuous math. The problem is even worse in cyclic systems, where the system vacillates between radically different states in a way that is completely ruled out of analytical game theory.

The ubiquity of cycles in formal models of elections is an extremely well-established result. This idea is actually one of the most ancient in formal political science, already appearing in the 18th century (de Condorcet, 1785). To get a clear idea of what an iterative election model might look like, let us first focus on the notion of sequence that is already present in the oldest result in the field, and then compare it to the types of iteration that I discuss in this paper.

Consider the cycle in the simplest form of Condorcet's Paradox, in which three electors ϕ_0, ϕ_1, ϕ_2 rank three candidates ξ_0, ξ_1, ξ_2 according to the rankings:

$\phi_0 : \xi_0 \succ \xi_1 \succ \xi_2$
$\phi_1 : \xi_1 \succ \xi_2 \succ \xi_0$
$\phi_2 : \xi_2 \succ \xi_0 \succ \xi_1$

de Condorcet (1785) imagines a sequence of pairwise contests between the three candidates: if ξ_0 faces off against ξ_1, then ξ_0 earns the votes of ϕ_0 and ϕ_2 and is victorious, but similarly ξ_1 would defeat ξ_2, and ξ_2 would defeat ξ_0. The famous result is that the group's expressed preference ranking of $\xi_0 \succ \xi_1 \succ \xi_2 \succ \xi_0$ is intransitive, even though each individual's preference ranking is transitive. Another way of stating the result is to say

[4]Italics are in the original document.

that an iterative cycle is present in the Condorcet system. If we were to hold an arbitrarily long sequence of pairwise contests and record which candidate won each time, then there would be no clear way of declaring one candidate to be the aggregate winner over time; in the many-contest limit, no candidate would win more pairwise contests than the other two. This is a simple cycle in a voting game. A closely related idea forms the foundation of the McKelvey-Schofield Chaos Theorem, which also concerns the possible cyclicities arising from a sequence of contests held across some decision space (McKelvey, 1976, Schofield, 1978).

The types of cycles that I deal with in this paper are motivated by the idea that cycles are completely unavoidable in election games, but with one crucial difference in the setup: unlike all of the previous social choice work that I know of, I am not concerned with cycles that arise from one election to the next, but rather with cycles that arise within the span of one election. We will impose a notion of iteration within one model of one election contest; in each iteration, electors signal their intention to vote for one of the candidates or to abstain, and then those signals are used by other electors to choose their best response.[5] When we use this iterative setup, we will find that cycles in the best response strategies of electors from one iteration to the next very often dominate within one election. This is an interesting social choice question in its own right, but in this paper I am narrowly focused on solving the specific methodological problem posed by the fact that these cycles are so ubiquitous: namely, I argue that this challenge is why voting games represent a useful test case for a broader methodology of how to define stopping conditions for iterative models of political processes.

Defining the equilibrium itself is only the first part of my methodological agenda in this paper. The other goal is to understand what types of patterns we might see when we stop the model. In the case that I investigate in this paper, the case of an iterative voting game, I am particularly interested in understanding what types of individual-level behaviours are possible given different patterns in the aggregate-level parameters. The central focus of the theorems in this paper will be to prove regularities in individual electors' strategies in a given iteration based on patterns in the corresponding aggregate vote totals. There is both a short-term methodological reason and a longer-term empirical reason that this is useful. Since the list of vote totals per candidate will always be much shorter than the list of individual vote choices made by electors (the former being a sum over the latter), in cases where the list of vote totals per candidate substantially constrains the possible strategies followed by electors, we may get very substantial details about the individual-level system for free simply by checking the list of candidates' vote totals. We will see that this is indeed a very common situation. Some theorems in this paper, for example Theorem 15, introduce regularities that could plausibly save billions of operations in real models by suggesting simpler ways of checking interesting details of the system. The long-term empirical motivation is that, in the real study of democratic elections, the prevalence of secret ballots and the sparsity of reliable data about individuals' behaviour means that electors' strategies are not really measurable; most of the high-quality data about elections is the candidates' real or expected vote totals. So, it will always be a useful empirical angle to ask what we know about the whole system

[5] I should also explicitly note that unlike de Condorcet (1785), I am not concerned with pairwise contests, because this is not how real elections are held – except of course in the case of elections which happen to only include 2 candidates, but even then electors have the third option of abstaining.

by observing just the candidates' vote totals.

In Section 2, I precisely define the system of interest, including two simple rulesets for electors to update their probabilities of being pivotal. I then use this system to illustrate an equilibrium idea which I define in Section 3, along with detailed examples of every type of equilibrium that I claim can exist. In Section 4 I state and prove a series of theorems about the behaviours of iterative voting games at equilibrium for both of the pivotal probability updating rules that I consider. Perhaps this paper's central result is the proof that, not only are cycles ubiquitous in voting games, but actually every iterative voting game which maps any aggregate vote total onto one unique aggregate vote total is cyclical (Theorem 7). To gain leverage on exactly what is possible at equilibrium, I prove that certain common simulation properties (like a particular pattern in the aggregate vote total) are incompatible with other common properties (like a particular pattern in the sequences of individual strategies), which dramatically simplifies the possible equilibria in iterative voting games (Theorem 8, Theorem 9, Theorems 13 through 17, Theorem 22, Theorem 24 and Theorem 25). I also prove some fundamental limitations on electors' behaviours in the system, for example that when electors have identical preference orderings and identical costs they can adopt different sequences of strategies at equilibrium, but only when utility values are not normalized (Theorem 10, Theorem 11, and Theorem 12). This provides a strong, and I believe novel, motivation for the convention of normalizing strategies between players. I close by discussing the problem of searching all of the possible equilibria in one formal election model, and by demonstrating the beginnings of a project to locate and usefully visualise those equilibria. A substantial methodological component of this paper is accompanying software like this; as well as proving the 26 Theorems that I state in this paper, I check and illustrate them with more than 11,000,000 runs of eight computer simulations that I wrote. These theorems establish regularities in iterative voting games and demonstrate why my equilibrium definition is a useful way of thinking about these models.

2 Definition of the iterative system

Consider a system of N electors $\{\phi_1, \phi_2, \ldots, \phi_N\}$ and M candidates $\{\xi_1, \xi_2, \ldots, \xi_M\}$.[6] Electors hold some preference over candidates, and their goal is to either choose the most effective candidate to support, or to abstain from voting. In every iteration, each elector i, $i \in \mathbb{N}, 1 \leq i \leq N$ holds the strategy set $S_i = \{s_1, s_2, \ldots, s_M, s_A\}$, where each s_j for $j \in \mathbb{N}, 1 \leq j \leq M$ corresponds to voting for candidate j, and s_A denotes abstaining. All of the electors we will consider are payoff-maximizers who wish to avoid two dangers: they do not want to waste their vote on a candidate who has an extremely small chance of winning the election, and they do not want to vote for any candidate if it would be better to abstain from voting. The candidates in these models are simply the objects of preferences; they will not make decisions in any of the models in this paper. For simplicity, all elections take place in a single district under a Single Non-Transferable Vote single-round plurality rule electoral system where the typical rules of democratic elections apply, so for example every electors'

[6] I unfortunately have to introduce a substantial amount of new notation in this paper. I therefore include a full glossary of notation in Section 6; every nonstandard symbol which is introduced in this paper is defined in the glossary.

vote is assumed to count equally.

I will consider iterative systems of payoff-maximizing agents.[7] First every elector declares an intention either to support some candidate or to abstain. By consulting the intentions stated by the other electors, which we should think of as being something like a public opinion poll that is conducted at the start of every single iteration, every elector calculates the payoff that they would expect from each possible action, using the classical formulation of subtracting the cost they would incur by voting from their expected utility of voting for some candidate. Here, the expected utility of voting for a candidate is the elector's probability of being pivotal in the election of that candidate times how much more payoff they would obtain from that candidate's victory than from some alternative (Downs, 1957, Riker and Ordeshook, 1968).[8] In this section, I will formalize this system.

Let $p_t^i(\xi_j, \xi_{j'})$ represent the following quite specific notion of pivotality: this is the probability that a vote by elector i could either create or break a tie between candidates j and j', and that candidates j and j' have more votes than any other candidates. For simplicity, I have compacted several distinct ideas into this one symbol: really, the act of breaking a tie should be more desirable than the act of creating a tie, and the probability that these candidates obtain higher totals than any other candidates is a separate event. I have combined these distinct ideas into one concise symbol because in this paper I do not aim to make any advancements in the logic of pivotal probabilities, except where I note a few simplifications that aid in their efficient calculation in the computer. The goal of this section is only to clarify the general setting of computational voting games that I consider, and to get some minimal believable notion of pivotality to use in the models I will investigate.

In iteration $t+1$, the expected utility for some elector i associated with voting for some candidate j, represented by the symbol $u^{i \to j}$, is given by the following system of equations:

$$u_{t+1}^{i \to 1} = p_t^i(\xi_1, \xi_2)(u^{i \to 1} - u^{i \to 2}) + p_t^i(\xi_1, \xi_3)(u^{i \to 1} - u^{i \to 3}) + \ldots + p_t^i(\xi_1, \xi_M)(u^{i \to 1} - u^{i \to M}) - c_i$$

[7] I assume rationality throughout this paper. In particular, I assume both that for any preference relation in an electorate I can freely produce a utility function which corresponds to that preference relation, and also that if an elector holds a particular utility function then it must reveal that they hold a corresponding preference relation. This might be an unexpected assumption for a paper about computational modeling, since there appears to be a solid consensus that one of the most promising features of computational models in political science is that they can accomodate non-rational players much more easily than traditional game theory can (Bendor et al., 2011). There are two major reasons to put that aside for the moment. First, this paper sets out to reduce the gap between game theory results and computational results; relaxing the crucial foundational assumptions of traditional game theory (von Neumann and Morgenstern, 1953) would risk turning a gap into a yawning chasm (Siegel, 2018). Second, because so many of the questions I engage with in this paper are almost completely open, as a practical matter I wish to remain as close as possible to a familiar framework while sorting out voting games' possible equilibrium states. I note however that if we did not assume rationality, so long as the agents are payoff-maximizing, everything I claim about utility functions should remain true for utility functions and everything I claim about preference relations should remain true about preference relations – except in the very few cases like Theorem 22 where I explicitly appeal to rationality – but conclusions about utility functions may not provide any information about preference relations and vice versa. So I believe that similar results would hold, up to a careful change of vocabulary.

[8] In this paper, I only consider systems in which every elector updates in every iteration. Certainly we can imagine interesting models in which electors do not update in every iteration – not every person pays attention to every poll. However, this is a dramatically more complicated case than the one I consider here, although I expect that many of the results which apply to a system in which not all electors update in each iteration will be similar, in the many-iteration limit, to the results I present in this paper.

$$u^{i\to 2}_{t+1} = p^i_t(\xi_2, \xi_1)(u^{i\to 2} - u^{i\to 1}) + p^i_t(\xi_2, \xi_3)(u^{i\to 2} - u^{i\to 3}) + \ldots + p^i_t(\xi_2, \xi_M)(u^{i\to 2} - u^{i\to M}) - c_i$$

$$\vdots$$

$$u^{i\to M}_{t+1} = p^i_t(\xi_M, \xi_1)(u^{i\to M} - u^{i\to 1}) + p^i_t(\xi_M, \xi_2)(u^{i\to M} - u^{i\to 2}) + \ldots + p^i_t(\xi_M, \xi_{M-1})(u^{i\to M} - u^{i\to M-1}) - c_i$$

In the above exposition of the system I have assumed that only two-way ties are possible, which I will do throughout this paper. I do so because, for nontrivial numbers of candidates, the relaxation of this assumption is combinatorially explosive, and I claim that the results I present will be invariant if larger ties are permitted because the probability of more than two candidates tying is always orders of magnitude smaller than the probability of two candidates tying. This simplification is common in the literature on pivotal voting logic.

Next, in the interest of stating the system as compactly as possible, this system of equations can be equivalently represented in matrix form, by adding the costs to both sides:

$$u^{i\to 1}_{t+1} + c_i = \begin{bmatrix} p^i_t(\xi_1, \xi_2) & p^i_t(\xi_1, \xi_3) & \ldots & p^i_t(\xi_1, \xi_M) \end{bmatrix} \begin{bmatrix} u^{i\to 1} - u^{i\to 2} \\ u^{i\to 1} - u^{i\to 3} \\ \vdots \\ u^{i\to 1} - u^{i\to M} \end{bmatrix}$$

$$u^{i\to 2}_{t+1} + c_i = \begin{bmatrix} p^i_t(\xi_2, \xi_1) & p^i_t(\xi_2, \xi_3) & \ldots & p^i_t(\xi_2, \xi_M) \end{bmatrix} \begin{bmatrix} u^{i\to 2} - u^{i\to 1} \\ u^{i\to 2} - u^{i\to 3} \\ \vdots \\ u^{i\to 2} - u^{i\to M} \end{bmatrix}$$

$$\vdots$$

$$u^{i\to M}_{t+1} + c_i = \begin{bmatrix} p^i_t(\xi_M, \xi_1) & p^i_t(\xi_M, \xi_2) & \ldots & p^i_t(\xi_M, \xi_{M-1}) \end{bmatrix} \begin{bmatrix} u^{i\to M} - u^{i\to 1} \\ u^{i\to M} - u^{i\to 2} \\ \vdots \\ u^{i\to M} - u^{i\to M-1} \end{bmatrix}$$

Now we can simplify the system by means of the following lemma:

Lemma 1. *Pivotal probability is symmetrical: for any elector ϕ_i and two candidates $\xi_j, \xi_{j'}$, $p^i(\xi_j, \xi_{j'}) = p^i(\xi_{j'}, \xi_j)$.*

Proof. The event in which an elector can create a first-place tie between two candidates and the event in which an elector can break a first-place tie between two candidates are mutually exclusive. Say that the probability of creating a first-place tie between candidates ξ_j and $\xi_{j'}$ is $g(\xi_j, \xi_{j'})$, and the probability of breaking a first-place tie between those candidates is $h(\xi_j, \xi_{j'})$. Because these events are mutually exclusive, and because they are the only situations in which the elector is pivotal, therefore the elector's probability of being pivotal in the victory of ξ_j over $\xi_{j'}$ is given by $p(\xi_j, \xi_{j'}) = g(\xi_j, \xi_{j'}) + h(\xi_j, \xi_{j'})$. Similarly, the elector's probability of being pivotal in the victory of $\xi_{j'}$ over ξ_j is given by $p(\xi_{j'}, \xi_j) = g(\xi_{j'}, \xi_j) + h(\xi_{j'}, \xi_j)$, with both probabilities conditional on these candidates' having more votes than all other candidates. Because the events in which ξ_j and $\xi_{j'}$ are either tied or tied up to ± 1 vote and ahead of all other candidates are identical situations whether the elector favours candidate ξ_j or candidate $\xi_{j'}$, we have $g(\xi_j, \xi_{j'}) = g(\xi_{j'}, \xi_j)$ and also $h(\xi_j, \xi_{j'}) = h(\xi_{j'}, \xi_j)$. Hence $p(\xi_j, \xi_{j'}) = p(\xi_{j'}, \xi_j)$. □

After constructing a matrix which contains all the pivotal probabilities for some elector, we can use Lemma 1 to observe a useful simplifying property in this matrix. For some arbitrary elector, construct that matrix as follows:

$$\mathbf{P} = \begin{bmatrix} p(\xi_1,\xi_1) & p(\xi_1,\xi_2) & p(\xi_1,\xi_3) & \dots & p(\xi_1,\xi_{M-1}) & p(\xi_1,\xi_M) \\ p(\xi_2,\xi_1) & p(\xi_2,\xi_2) & p(\xi_2,\xi_3) & \dots & p(\xi_2,\xi_{M-1}) & p(\xi_2,\xi_M) \\ p(\xi_3,\xi_1) & p(\xi_3,\xi_2) & p(\xi_3,\xi_3) & \dots & p(\xi_3,\xi_{M-1}) & p(\xi_3,\xi_M) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p(\xi_{M-1},\xi_1) & p(\xi_{M-1},\xi_2) & p(\xi_{M-1},\xi_3) & \dots & p(\xi_{M-1},\xi_{M-1}) & p(\xi_{M-1},\xi_M) \\ p(\xi_M,\xi_1) & p(\xi_M,\xi_2) & p(\xi_M,\xi_3) & \dots & p(\xi_M,\xi_{M-1}) & p(\xi_M,\xi_M) \end{bmatrix}$$

Now observe that since candidates cannot be in a tie or a near-tie with themselves, the set of events satisfying this condition is empty. Since it is an elementary property of probabilities that $\mathbb{P}(\emptyset) = 0$, therefore for any arbitrary candidate j, $p(\xi_j, \xi_j) = 0$. So the pivotality matrix reduces to:

$$\mathbf{P} = \begin{bmatrix} 0 & p(\xi_1,\xi_2) & p(\xi_1,\xi_3) & \dots & p(\xi_1,\xi_{M-1}) & p(\xi_1,\xi_M) \\ p(\xi_2,\xi_1) & 0 & p(\xi_2,\xi_3) & \dots & p(\xi_2,\xi_{M-1}) & p(\xi_2,\xi_M) \\ p(\xi_3,\xi_1) & p(\xi_3,\xi_2) & 0 & \dots & p(\xi_3,\xi_{M-1}) & p(\xi_3,\xi_M) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p(\xi_{M-1},\xi_1) & p(\xi_{M-1},\xi_2) & p(\xi_{M-1},\xi_3) & \dots & 0 & p(\xi_{M-1},\xi_M) \\ p(\xi_M,\xi_1) & p(\xi_M,\xi_2) & p(\xi_M,\xi_3) & \dots & p(\xi_M,\xi_{M-1}) & 0 \end{bmatrix}$$

And by Lemma 1:

$$\mathbf{P} = \begin{bmatrix} 0 & p(\xi_1,\xi_2) & p(\xi_1,\xi_3) & \dots & p(\xi_1,\xi_{M-1}) & p(\xi_1,\xi_M) \\ p(\xi_1,\xi_2) & 0 & p(\xi_2,\xi_3) & \dots & p(\xi_2,\xi_{M-1}) & p(\xi_2,\xi_M) \\ p(\xi_1,\xi_3) & p(\xi_2,\xi_3) & 0 & \dots & p(\xi_3,\xi_{M-1}) & p(\xi_3,\xi_M) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p(\xi_1,\xi_{M-1}) & p(\xi_2,\xi_{M-1}) & p(\xi_3,\xi_{M-1}) & \dots & 0 & p(\xi_{M-1},\xi_M) \\ p(\xi_1,\xi_M) & p(\xi_2,\xi_M) & p(\xi_3,\xi_M) & \dots & p(\xi_{M-1},\xi_M) & 0 \end{bmatrix}$$

A pattern emerges which has a simple formalization:

Theorem 2. $\mathbf{P}$ *is symmetric.*

Proof. $\forall j, j' \in \{1; M\}, \mathbf{P}_{j,j'} = p(\xi_j, \xi_{j'})$, so by Lemma 1, $\mathbf{P}_{j,j'} = \mathbf{P}_{j',j}$. ☐

Now, we can make a similarly simple statement about the matrix $\mathbf{U}$ which contains of all of the sincere utilities for an arbitrary elector i:

$$\mathbf{U} = \begin{bmatrix} u^{i\to 1} - u^{i\to 1} & u^{i\to 2} - u^{i\to 1} & u^{i\to 3} - u^{i\to 1} & \dots & u^{i\to M-1} - u^{i\to 1} & u^{i\to M} - u^{i\to 1} \\ u^{i\to 1} - u^{i\to 2} & u^{i\to 2} - u^{i\to 2} & u^{i\to 3} - u^{i\to 2} & \dots & u^{i\to M-1} - u^{i\to 2} & u^{i\to M} - u^{i\to 2} \\ u^{i\to 1} - u^{i\to 3} & u^{i\to 2} - u^{i\to 3} & u^{i\to 3} - u^{i\to 3} & \dots & u^{i\to M-1} - u^{i\to 3} & u^{i\to M} - u^{i\to 3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u^{i\to 1} - u^{i\to M-1} & u^{i\to 2} - u^{i\to M-1} & u^{i\to 3} - u^{i\to M-1} & \dots & u^{i\to M-1} - u^{i\to M-1} & u^{i\to M} - u^{i\to M-1} \\ u^{i\to 1} - u^{i\to M} & u^{i\to 2} - u^{i\to M} & u^{i\to 3} - u^{i\to M} & \dots & u^{i\to M-1} - u^{i\to M} & u^{i\to M} - u^{i\to M} \end{bmatrix}$$

Simplifying,

$$\mathbf{U} = \begin{bmatrix} 0 & u^{i\to 2} - u^{i\to 1} & u^{i\to 3} - u^{i\to 1} & \dots & u^{i\to M-1} - u^{i\to 1} & u^{i\to M} - u^{i\to 1} \\ u^{i\to 1} - u^{i\to 2} & 0 & u^{i\to 3} - u^{i\to 2} & \dots & u^{i\to M-1} - u^{i\to 2} & u^{i\to M} - u^{i\to 2} \\ u^{i\to 1} - u^{i\to 3} & u^{i\to 2} - u^{i\to 3} & 0 & \dots & u^{i\to M-1} - u^{i\to 3} & u^{i\to M} - u^{i\to 3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u^{i\to 1} - u^{i\to M-1} & u^{i\to 2} - u^{i\to M-1} & u^{i\to 3} - u^{i\to M-1} & \dots & 0 & u^{i\to M} - u^{i\to M-1} \\ u^{i\to 1} - u^{i\to M} & u^{i\to 2} - u^{i\to M} & u^{i\to 3} - u^{i\to M} & \dots & u^{i\to M-1} - u^{i\to M} & 0 \end{bmatrix}$$

And a property emerges which is neatly analogous to the last simplifcation of the pivotality matrix:

$$\mathbf{U} = \begin{bmatrix} 0 & u^{i\to 2} - u^{i\to 1} & u^{i\to 3} - u^{i\to 1} & \dots & u^{i\to M-1} - u^{i\to 1} & u^{i\to M} - u^{i\to 1} \\ -(u^{i\to 2} - u^{i\to 1}) & 0 & -u^{i\to 3} - u^{i\to 2} & \dots & u^{i\to M-1} - u^{i\to 2} & u^{i\to M} - u^{i\to 2} \\ -(u^{i\to 3} - u^{i\to 1}) & -(u^{i\to 2} - u^{i\to 3}) & 0 & \dots & u^{i\to M-1} - u^{i\to 3} & u^{i\to M} - u^{i\to 3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -(u^{i\to M-1} - u^{i\to 1}) & -(u^{i\to M-1} - u^{i\to 2}) & -(u^{i\to M-1} - u^{i\to 3}) & \dots & 0 & u^{i\to M} - u^{i\to M-1} \\ -(u^{i\to M} - u^{i\to 1}) & -(u^{i\to M} - u^{i\to 2}) & -(u^{i\to M} - u^{i\to 3}) & \dots & -(u^{i\to M} - u^{i\to M-1}) & 0 \end{bmatrix}$$

Theorem 3. $\mathbf{U}$ *is skew-symmetric.*

Proof. $\forall j, j' \in \{1; M\}$, and some elector ϕ_i, $\mathbf{U}_{j,j'} = u^{i\to j'} - u^{i\to j}$. With $u^{i\to j'} - u^{i\to j} = -(u^{i\to j} - u^{i\to j'})$, therefore $\mathbf{U}_{j,j'} = -\mathbf{U}_{j',j}$. ☐

The properties identified in Theorem 2 and Theorem 3 have two practical implications. First, in any algorithm which calculates these matrices, they halve the time complexity of the step which computes the matrix. Second, they give hints as to which matrix operations will be robust and efficient when applied to pivotality and utility matrices.

To finish stating the system, next define the vector of an elector's expected utilities from each possible pure strategy, $\mathbf{E}$, as

$$\mathbf{E} = \begin{bmatrix} u^{i\to 1} \\ u^{i\to 2} \\ u^{i\to 3} \\ \vdots \\ u^{i\to M-1} \\ u^{i\to M} \\ u^{i\to A} \end{bmatrix}$$

where the last element in this vector corresponds to the utility that the elector obtains not by voting for any candidate, but rather by abstaining. Notice that since in the conventional voting logic of Riker and Ordeshook (1968) we typically subtract the cost from the sincere utility associated with a given candidate, it holds that the ordering of utilities by magnitude is invariant under addition by this constant (with the conventional assumption that casting a vote for any two different candidates carries an identical cost, and abstaining incurs both cost and benefit 0). We can therefore add the cost value c_i for elector i to every utility, so that the utility associated with voting for candidate j is simply

$$u^{i\to j} - c_i + c_i = u^{i\to j}$$

and it is preferred over abstaining if

$$u^{i\to j} > u^{i\to A} + c_i$$

so if

$$u^{i\to j} > c_i$$

Now it is possible to state the system concisely. In any iteration t, each elector ϕ_i calculates the expected utilities corresponding to their pure strategies as:

$$\mathbf{E}_{t+1} = [\text{diag}(\mathbf{P}_t\mathbf{U}_t), c_i]$$

where the last element in this list is the cost associated with voting (now also understood as the benefit of abstaining), c_i. Then we can simplify this slightly with the following observation:

Theorem 4. *For any elector, their pivotality matrix* $\mathbf{P}$ *may change from one iteration to the next, but their matrix of sincere utilities* $\mathbf{U}$ *can not.*

Proof. The claim follows directly from the definition of sincere utilities. □

So, the system above simplifies to

$$\mathbf{E}_{t+1} = [\text{diag}(\mathbf{P}_t\mathbf{U}), c_i] \tag{1}$$

Theorem 5. *The pure strategy best response for elector ϕ_i in iteration $t+1$ is to vote for candidate ξ_j, where $j = argmax([diag(\mathbf{P}_t\mathbf{U}), c_i])$, if that maximal element is any but the last element, and to abstain if the maximal element is the last element.*

Proof. By Equation 1, $\text{argmax}([\text{diag}(\mathbf{P}_t\mathbf{U}), c_i])$ returns the strategy which corresponds to the largest value of $\mathbf{E}_{t+1}$. □

Equation 1 is a statement of how one elector updates their expected utilities from one iteration to the next. Since the pivotality matrix $\mathbf{P}$ changes from one iteration to the next whereas the matrix of sincere utilities $\mathbf{U}$ remains fixed throughout the model for a given elector, in order to fully capture the iterative system we still need to specify how exactly the pivotality matrix updates. Each element in the pivotality matrix is a function only of the probability that all other electors in the system will create or be within one vote of creating a first-place tie between two candidates, so the pivotality matrix is a function of the aggregate vote totals. So it remains only to specify how the aggregate totals update from each iteration to the next.

We want a compact representation of a decision vector that states whether or not some elector ϕ_i chooses each pure strategy. So, if there are 5 candidates and the optimal strategy is to play s_3 and cast a vote for candidate ξ_3, we want to return the vector $[0, 0, 1, 0, 0, 0]$. A useful notational tool here is Iverson Brackets, $\Big[\cdot\Big]$, which map a proposition onto the number 1 if that proposition is true and onto the number 0 if that proposition is false. So, construct a vector which holds a 1 in position j if the argmax of the expected utilities is strategy s_j or in the final position if the maximal expected utility is gained by abstaining s_A, and a 0 in all other indices:[9]

$$\mathbf{e}^i_{t+1} = \Big[j = \text{argmax}([\text{diag}(\mathbf{P}^i_t\mathbf{U}^i), c_i])\Big] \tag{2}$$

Let us now make the simplifying assumption that the vector $\mathbf{e}$ returned by Equation 2 always returns exactly one element equal to 1 and all other elements equal 0. This means that there are no ties in the maximum expected utility among pure strategies. The case in which electors have equal maximal expected utility from randomizing over multiple pure strategies is the case in which electors play mixed strategies, which I set aside for the moment simply to gain mathematical traction on the iterative system. Of course this case is a crucial area to explore once we have made progress in understanding the pure strategy iterative system.

[9] I use $\mathbf{e}$ specifically to invite the comparison between this vector and a unit vector, where the preferred candidate (or abstention) corresponds to the unit vector of one dimension in an $M+1$-dimensional space.

Speaking only about pure strategies, we can now observe that payoff-maximizing electors will produce the following aggregate vote total in every iteration:

$$\mathbf{V}_{t+1} = \sum_{i=1}^{N} \left[j = \text{argmax}([\text{diag}(\mathbf{P}_t^i \mathbf{U}^i), c_i]) \right] \tag{3}$$

So, stated simply using Equation 2,

$$\mathbf{V}_{t+1} = \sum_{i=1}^{N} \mathbf{e}_i \tag{4}$$

This vector $\mathbf{V}$ is the one which informs the pivotality matrix $\mathbf{P}$ for each elector. One final complication is that each elector $\phi_{i'}$ must sum over every elector other than themselves, because in calculating pivotality they are trying to decide how to vote based on the expected behaviour of every other elector in the system. So, the pivotality matrix $\mathbf{P}_{t+1}^{i'}$ of elector $\phi_{i'}$ in iteration $t+1$ actually takes a specific vector of votes $\mathbf{V}_{t+1}^{i'}$ which only considers electors other than elector $\phi_{i'}$. This vector is obtained by:

$$\mathbf{V}_{t+1}^{i'} = \sum_{\substack{i=1 \\ i \neq i'}}^{N} \mathbf{e}_i \tag{5}$$

From this explanation so far there is now only one major component missing: what is the ruleset by which electors translate aggregate vote totals into a pivotal probability? In this paper I will consider two rulesets.

Definition 1. *Naïve Pivotality: This pivotality rule is designed to be the simplest possible pivotality update. Every elector simply assumes that whatever happened in the previous iteration is certain to be the final result. In a given iteration, an elector surveys the state of the world by checking whether or not any two candidates were tied or within one vote of being tied in the previous iteration (throughout this paper I will refer to this state as "a tie or a tie* ± 1*"). If there was a tie or a tie* ± 1*, that elector takes their set of strategic utilities for the strategies corresponding to each candidate involved in the tie or the tie* ± 1 *to be 1 times their set of sincere utilities – that is, they naïvely assume that the tie or tie* ± 1 *that they just observed will certainly be the final election result, and they assume that they will obtain their sincere utility for creating or breaking that tie. For all candidates that were not involved in a tie or a tie* ± 1 *in the previous iteration, every elector takes their strategic utility for voting for that candidate to be 0 times their sincere utility; that is, they assume that this candidate will certainly not be involved in a tie or a tie* ± 1*, so they will obtain 0 utility from voting for them. Since this rule does not specify behaviour in iteration 0, we will assume throughout the paper that all electors arbitrarily begin by naïvely intending to vote, and then we will explore variations in this starting seed in Section 4.4.*

Definition 1 is designed to be as simple as possible, so that we can get a handle on the effects of iterating even an incredibly simple ruleset. However, it is not connected to any of the literature on pivotal voting logic; it is not a reasonable model of how electors should calculate their best responses, and is only useful to get mathematical traction over iterative voting games. So, in this paper I will also study a reasonable but more complicated updating ruleset:

Definition 2. *Skellam Pivotality: Electors' probability of being pivotal is drawn from the Skellam distribution. This is situated in the large Poisson voting games framework developed by Myerson (2000). Here, that framework replaces the typical assumption in analytical game theory that the population size is exactly known and can be calculated using the full multinomial tie probabilities with the much more computationally feasible assumption that electors receive some signal μ about the size of the distribution, which I draw from a uniform distribution over a small range, and take that value to be a Poisson distributed random variable with mean N. Observing only the proportion of the population that will vote for each candidate, the elector can obtain a pivotal probability by estimating the expected distribution of relative vote counts across all possible combinations of candidates. The full logic and exact calculations connecting this to the distribution in Skellam (1946) are provided in Baltz et al. (2018: eq. 6 and eq. 7). Because advancements in pivotal probability calculations are not of central interest to this paper, I have simply constructed all of the Skellam Pivotality models in this paper to exactly match the logic laid out there.*

The idea of writing down several different ways of translating vote totals into a pivotality value might be unsatisfying; in particular, it might appear that there is a unique method that will provide a best response. It is crucial to note here that this is not a process of finding a best response strategy; really what we are doing here is specifying just how sophisticated or limited electors are in the guesses they form. This should be thought of as specifying the extent to which electors are accurate about what behaviours to expect from other members of their community, and based on these expectations they will unerringly calculate best responses. I discuss this further in subsequent sections of this paper.

This completes the specification of the full dynamical system in pure strategies, which is summarized by the following procedure:

↪ Seed the system arbitrarily, say with sincere voting
↪ Every iteration, all electors see a poll or census of the expected vote count
↪ Electors calculate their expected pivotal probability using those aggregate values
↪ Electors calculate their expected utility from each pure strategy
↪ Electors signal the best response among their strategies, which will constitute the next iteration's poll

Finally, we need to make a crucial distinction between two types of systems of interest.

Definition 3. *A **deterministic voting game** is an iterative election model which maps any aggregate vote count $\mathbf{V}_t$ onto exactly one aggregate vote count $\mathbf{V}_{t+1}$ in the next iteration.*

Definition 4. *A* ***probabilistic voting game*** *is an iterative election model which can with positive probability map some aggregate vote count* $\mathbf{V}_t$ *onto any one of several different aggregate vote counts in the next iteration.*

Naïve Pivotality models are an example of deterministic voting games. Once each elector has guessed the size of the population, Skellam Pivotality models are also an example of deterministic voting games, since the pivotal probability values are subsequently obtained by a calculation of the PMF and CDF of the Skellam distribution (Baltz et al., 2018). The main situation in which probabilistic voting games arise is when electors are able to select mixed strategies.

With the systems of interest completely specified, we can now define what constitutes an equilibrium in such a model.

3 Equilibria

In this section, I will first define what I mean by "equilibrium", and then begin to state results about what is required for the different types of equilibria to exist.

3.1 Equilibrium definition

In iterative computational models of political processes, I know of only three other explicit definitions of "equilibrium": one given by Laver and Sergenti (2012), one by Bendor et al. (2011), and another by Siegel (2018). In Laver and Sergenti (2012) and Bendor et al. (2011), the equilibrium notion is only briefly discussed, because these works are primarily describing applications of computational models. In Siegel (2018), the equilibrium definition is one comparatively short section in a paper that is mainly devoted to the problem of deriving comparative statics in computational models. The three definitions offered in these works are very similar: all of these authors imagine computational formal models as discrete state stochastic processes in which the transition from one iteration to the next can be modeled by markov chain-like transition probabilities, and their notions of equilibrium are stated in terms of the convergence of these transition probabilities over time (Bendor et al., 2011: p. 28; Laver and Sergenti, 2012: p. 64; Siegel, 2018).

I propose a two-part equilibrium definition, which I call an "iterative equilibrium". It is designed to classify two types of processes as equilibria. The first part categorizes as an equilibrium any situation in which a sequence of states may be distinct, but only by an arbitrarily small amount. This retains the central virtue of previous definitions by recognizing a system as stable even if the system has not completely settled down to one specific set of values which it will never deviate from, but it takes advantage of the existence of substantively meaningful parameters in the model by requiring that the equilibrium states must be sufficiently similar to one another. The second part explicitly categorizes repeating cycles as equilibria.

The definition will be stated in terms of the collection of parameters that comprise a voting game; these parameters exist both at the individual level and the aggregate level. An example of an individual-level parameter is an elector's expected utility from picking a particular strategy in some iteration, while an example of an aggregate-level parameter is the number of electors pursuing each strategy in a given iteration, which corresponds to the sum of the votes that a given candidate expects to receive. It is entirely possible for one parameter to settle down and another one not to, and we may or may not want to stop the model when one parameter settles down, which is why I state my equilibrium definition entirely in terms of individual parameters. That definition is as follows:

Definition 5. *A parameter ρ_t in an iterative voting game $\mathcal{M}$ reaches an* ***iterative equilibrium*** *in iteration t_τ if either of the following two conditions holds:*
i) $\exists \varepsilon$ sufficiently small so that $\left|\rho_{t'} - \rho_{t_\tau}\right| < \varepsilon \; \forall t'$ with $t' > t_\tau$.
ii) $\exists t_\tau$ so that, $\forall t' > t_\tau$, it holds that $\forall t'' > t'$, $\exists k$ so $t'' - t' \equiv 0 \mod k \implies \rho_{t''} = \rho_{t'}$.
Call an equilibrium of type i "static". Call an equilibrium of type ii k-cyclic or k-periodic.

Not all models hit equilibrium, although we will see in Theorem 7 that huge classes of models that we care about certainly produce a sequence that will be classified as an equilibrium under Definition 5. I will therefore use the word "end-state" to refer to any situation which might hold for all sufficiently advanced iterations of some iterative voting game, but without necessarily satisfying Definition 5. According to my nomenclature all equilibria are end-states, but not all end-states are equilibria. This distinction will be investigated in the following section.

3.2 Characterising the possible end-states

Claim 6. *Consider some population of electors $\phi_1, \cdots, \phi_N$. There are three possible end-states that an individual elector can reach. These end-states are:*
(1) The elector selects strategy s^*, and continues to pick s^* for each subsequent iteration.
(2) Across k iterations, $k \in \mathbb{N}$ so $k > 1$, the elector adopts each of k strategies, and in the $k+1$st iteration returns to the first strategy in the sequence. So, they select $s_1 \to \cdots \to s_k \to s_1 \to \cdots$. I call this sequence a "k-cycle", and in keeping with standard nomenclature in discrete dynamical systems, I call the system itself "k-periodic". Note that any k-cycler is also an l-cycler, where $l = nk$ for some $n \in \mathbb{N}$; a 2-cycler is also a 4-cycler and a 6-cycler, a 3-cycler is also a 6-cycler and a 9-cycler, and so on.
(3) The elector adopts a non-repeating sequence of strategies $s_1 \to s_2 \to \cdots$. Notice that such a sequence can, problematically, include any number of k-cycles, so long as the entire sequence is not composed of k-cycles.

Proof. Options (2) and (3) are a complete partition of the space of possible events; either the elector repeats or it does not. Option (1) is a subset of option (2). □

Using Definition 3, we can immediately identify a simplifying theorem.

Theorem 7. *Any deterministic voting game is k-periodic, for some $k \in \mathbb{N}$, where k is at most the number of distinct values that the aggregate vote counts can possibly adopt plus one.*

Proof. Enumerate the r possible values that the aggregate vote counts could adopt, and recall that because the system is deterministic, every such state necessarily leads to exactly one other state. Now consider the first r elements of the sequence of aggregate vote totals produced by any run of the model. Either each element appears exactly once within the first r elements, or it is not. First suppose that there exists some aggregate vote total $\mathbf{V}$ which appears multiple times within the first r elements of the sequence of aggregate vote totals. Because the system is deterministic, $\mathbf{V}$ must map with certainty onto some aggregate vote total $\mathbf{V}'$, which must map with certainty onto some aggregate vote total $\mathbf{V}''$, and so on. However, this sequence must be of length less than r, because otherwise there is a contradiction with the assumption that $\mathbf{V}$ appears multiple times within the first r elements of the sequence. Therefore, there is a cycle in the sequence of aggregate vote totals of length less than r. So, the system is periodic with period less than r. Next, suppose that there does not exist any element which appears multiple times within the first r elements. Because the system is deterministic, note that the last element $\mathbf{V}_r$ must map onto some aggregate vote total. Because the system is known to have length r, it can only be the case that $\mathbf{V}_r$ maps onto an element which has appeared previously in the sequence. This creates a repeating sequence of at most length $r+1$. Since one of these situations must be true, any deterministic voting game is at most $r+1$-periodic. □

This theorem simplifies the problem of studying the equilibria of these games. It is a useful result that any deterministic system, such as the Naïve Pivotality system, will necessarily reach an identifiable equilibrium, with a known upper bound on how many iterations it will take to repeat (although this upper bound is extremely large). Next, I will work through a typology of the possible aggregate end-states. Recalling the labels assigned to each possible end-state that an individual elector can adopt, where ① is a stable sequence, ② is a deterministic cycle, and ③ is disequilibrium, there are 7 possible combinations of types of electors that one electorate could contain:

$$\{①\}, \{①,②\}, \{①,③\}, \{②\}, \{②,③\}, \{③\}, \{①,②,③\}$$

But can all of these types actually coexist in one electorate, and if so, how? Let us examine each possible case individually. Call x, y, z arbitrary proportions, $x, y, z \in \mathbb{Q}, x, y, z \in [0; 1]$. Call the number of electors in category ①, the static electors, σ. Call the number of electors in category ②, so those undergoing k-cycles, κ. And call the number of electors in category ③, so those which never reach a static or cycling equilibrium, η. In all of these examples, unless otherwise stated, I will assume that all electors employ the Naïve Pivotality ruleset, for the purpose of simple exposition.

Start with the case in which $N = \sigma$, so the population is composed entirely of electors that settle down to one static strategy. This is the most comfortable scenario, because it is directly analogous to static-time game theory, in the way identified by Siegel (2018).

Example 1. *All electors are static*

2 electors ϕ_1, ϕ_2 with arbitrarily small positive costs c_1, c_2 choose between voting for one of two candidates ξ_1, ξ_2 or abstention A. Elector ϕ_1 has $u_1(\xi_1) > u_1(\xi_2)$ and $u_1(\xi_1) > c_1$, while elector 2 has $u_2(\xi_2) > u_2(\xi_1)$ and $u_2(\xi_2) > c_2$. Then the vote proceeds as follows:

Iteration 0

In dynamical systems terms, we need to pick an arbitrary seed for the model. In game theoretic terms, electors can just as reasonably pick any belief about the state of the world. Consider the initial strategy in which all electors vote for their sincere preference. The vector of vote totals is as follows, continuing the convention established in Equation 3 of listing the candidates in increasing order by index and with the total for abstention in the final position: $\mathbf{V}_0 = [1, 1, 0]$.

Iteration 1

$\hookrightarrow$ ϕ_1 Observes that if they had not voted for ξ_1, they would have failed to create a first-place tie for their preferred candidate. Since $u_1(\xi_1) > c_1$, ϕ_1 will vote.
$\hookrightarrow$ ϕ_2 Observes that if they had not voted for ξ_2, they would have failed to create a first-place tie for their preferred candidate. Since $u_2(\xi_2) > c_2$, ϕ_2 will vote.
The vote vector in this iteration is therefore $\mathbf{V}_1 = [1, 1, 0]$.

All subsequent iterations proceed identically; with certainty, the system maps the sequence $[1, 1, 0] \rightarrow [1, 1, 0] \rightarrow \cdots$. This is therefore a case where $N = \sigma$.

Next consider the case in which $N = x\sigma + y\kappa$, so some electors are cycling with $k > 1$ and others are static. This case appears very frequently in simple deterministic models.

Example 2. *Both multi-iteration and single-iteration cyclers*

Four electors, $\phi_1, \phi_2, \phi_3,$ and ϕ_4 hold the following sincere utilities related to two candidates ξ_1 and ξ_2:

$$\mathbf{u}^1 = \begin{bmatrix} u(\xi_1) = 1 \\ u(\xi_2) = 0 \end{bmatrix}$$
$$\mathbf{u}^2 = \begin{bmatrix} u(\xi_1) = 1 \\ u(\xi_2) = 0 \end{bmatrix}$$
$$\mathbf{u}^3 = \begin{bmatrix} u(\xi_1) = 1 \\ u(\xi_2) = 0 \end{bmatrix}$$
$$\mathbf{u}^4 = \begin{bmatrix} u(\xi_2) = 1 \\ u(\xi_1) = 0 \end{bmatrix}$$

Now suppose that elector ϕ_1 has a negative cost to voting, $c_1 < 0$ whereas all other electors $\phi_2, \phi_3,$ and ϕ_4 have costs to voting which are less than the utility they obtain from the victory of their top-ranked candidate, but greater than the 0 utility which we always assume electors obtain from abstaining, so $1 > c_2, c_3, c_4 > 0$. Suppose that strategic utilities are calculated according to the Naïve Pivotality method from Definition 1. Then we have the following sequential calculations.

Iteration 0

Electors are instantiated with their sincere utilities, and suppose that we seed the system by having every elector arbitrarily select their sincerely top-ranked preference. ξ_1 receives 3 votes and ξ_2 receives 1, yielding a vote vector of $\mathbf{V}_0 = [3, 1, 0]$.

Iteration 1

ϕ_1 does not expect to be pivotal, but because $c_1 < 0$, they nevertheless expect a higher payoff from voting than from abstaining, so they will continue to vote for their sincerely preferred candidate. All other electors ϕ_2, ϕ_3 and ϕ_4 observe that they would not have been pivotal in creating or breaking a tie for their preferred candidate in the previous iteration, so in this iteration voting provides $u_1^2 = u_1^3 = u_1^4 = 0$. Since $c_2, c_3, c_4 > 0$, all of $\phi_2, \phi_3,$ and ϕ_4 abstain from voting. So the vote vector becomes $\mathbf{V}_1 = [1, 0, 3]$.

Iteration 2

ϕ_1 votes. ϕ_4 observes that they would have been pivotal if they had voted, setting $u_2^4 = 1$, whereas ϕ_1 and ϕ_2 observe that they would not have been, setting $u_1^2 = u_1^3 = 0$. Since $0 < c_2, c_3, c_4 < 1$, ϕ_2 and ϕ_3 abstain but ϕ_4 casts a vote. The result is $\mathbf{V}_2 = [1, 1, 2]$.

Iteration 3

ϕ_1 votes. ϕ_4 observes that they were pivotal in causing their most-preferred candidate to tie with their least-preferred candidate, so their expected utility from voting is $u_3^4 = 1$. ϕ_2 and ϕ_3 observe that by voting, they could have broken the tie between their most-preferred and least-preferred candidates, so their expected utility from voting is now $u_3^2 = u_3^3 = 1$. So the vote vector in this iteration is $\mathbf{V}_3 = [3, 1, 0]$.

From this iteration onwards, the cycle repeats. For any iteration $t \in \mathbb{N}$, the following pattern holds, where v_1 is the number of votes for ξ_1, v_2 is the number of votes for ξ_2, and v_A is the number of electors who abstain:

$$t \equiv 0 \pmod 3 \implies \{v_1 = 3, v_2 = 1, v_A = 0\}$$

$$t \equiv 1 \pmod 3 \implies \{v_1 = 1, v_2 = 0, v_A = 3\}$$

$$t \equiv 2 \pmod 3 \implies \{v_1 = 1, v_2 = 1, v_A = 2\}$$

The aggregate behaviour in this system is therefore precisely 3-periodic, with 3 electors whose behaviour repeats in cycle of length 3, and with one elector whose behaviour repeats in a cycle of length 1.

Next consider the case in which $N = x\sigma + z\eta$. This can be obtained quite simply.

Example 3. *Stable and non-repeating*

Consider an electorate composed of two electors. One elector ϕ_1 chooses between candidates ξ_1, ξ_2 and abstention A in each iteration by sampling a value $u_t \sim \mathcal{U}(0;1)$, and voting for ξ_1 iff $0 \leq u_t < \frac{1}{3}$, voting for ξ_2 iff $\frac{1}{3} \leq u_t < \frac{2}{3}$, and abstaining iff $\frac{2}{3} \leq u_t \leq 1$. The other elector ϕ_2 has cost $c_2 > u^{2 \to 1}$ and $c_2 > u^{2 \to 2}$, so in every iteration ϕ_2 will abstain. Then ϕ_1 will neither cycle nor pick the same strategy every iteration, and ϕ_2 remains stable in every iteration.

Next consider the case in which $N = \kappa$. There are two major complications in this case which dramatically change the characteristics of the equilibrium. First, electors could cycle with different periods; at first it might appear that we could imagine cases where an electorate contains a mix of 2-cyclers and 3-cyclers. I will state several simplifying theorems about this possibility in Section 4. Second, not all electors need to be synchronized; we could imagine a case where one elector is static between iterations t and $t+1$ but then changes strategies between $t+1$ and $t+2$, while another elector changes strategies between iterations t and $t+1$ but then is static between $t+1$ and $t+2$. I will show conditions under which this is possible in Section 4. It might also appear that in this latter case, the aggregate equilibrium could remain constant even though the individuals in the system are forever cycling. The impossibility of this conjecture will be proved in Theorem 16 for any deterministic voting game.

What follows is an example of a system where the electors have equal periodicity and are synchronized in their cycle.

Example 4. *Exactly synchronized 2-cyclers*

4 electors $\phi_1, \phi_2, \phi_3, \phi_4$ which have positive costs c_1, c_2, c_3, c_4 choose between voting for one of two candidates ξ_1, ξ_2 or abstention A. ϕ_1 has $u_1(\xi_1) > u_1(\xi_2)$ and $u_1(\xi_1) > c_1$,

while ϕ_i for $i \in \{2,3,4\}$ have $u_i(\xi_2) > u_i(\xi_1)$ and $u_i(\xi_2) > c_i$. Then the vote proceeds as follows:

Iteration 0

Electors can choose any belief about the state of the world, so one possible consistent choice is for all electors to vote. Then ξ_1 receives 1 vote while ξ_2 receives 3 votes, with 0 electors abstanining, so $\mathbf{V}_0 = [1,3,0]$.

Iteration 1

Every elector individually notices that they neither created nor broke a desirable first-place tie. Since all electors have positive costs but now expect 0 utility from voting, all four electors abstain, so $\mathbf{V}_1 = [0,0,4]$.

Iteration 2

Every elector individually notices that by voting they could have broken a first-place tie, so ξ_1 receives 1 vote while ξ_2 receives 3 votes, with 0 electors abstanining, yielding $\mathbf{V}_2 = [1,3,0]$.

This pattern repeats in every subsequent iteration, with each elector in a synchronized 2-cycle. The following pattern holds:

$$t \equiv 0 \pmod 2 \implies \{v_1 = 1, v_2 = 3, v_A = 0\}$$
$$t \equiv 1 \pmod 2 \implies \{v_1 = 0, v_2 = 0, v_A = 4\}$$

Next consider the case in which $N = y\kappa + z\eta$. This is an extremely counterintuitive possibility; if there is a random actor creating unpredictable noise in the system, then how could that system possibly include electors who cycle through a sequence of strategies with certainty? Consider the following example.

Example 5. *Cycling electors in a non-cyclic electorate*

6 electors $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6$ which have positive costs $c_1, c_2, c_3, c_4, c_5, c_6$ choose between voting for one of two candidates ξ_1, ξ_2 or abstention A. ϕ_1 has $u_1(\xi_2) > u_1(\xi_1)$ and $u_1(\xi_2) > c_1$, while ϕ_i for $i \in \{2,3,4,5,6\}$ have $u_i(\xi_1) > u_i(\xi_2)$ and $u_i(\xi_1) > c_i$. Now

suppose that ϕ_6 has some nonzero probability of trembling and playing a move other than their best response in any iteration. So, for every alternative strategy j other than the best response strategy of ϕ_6, there is a probability $q_t^j > 0$ that ϕ_6 selects that strategy instead of the best response strategy. Then the vote proceeds as follows:

Iteration 0

Electors can choose any belief about the state of the world, so one possible consistent choice is for all electors to vote. Say that ϕ_6 may tremble in this first round. Then with probability q_0^1 we have $\mathbf{V}_0^1 = [4, 1, 1]$, with probability q_0^2 we have $\mathbf{V}_0^2 = [4, 2, 0]$, and with probability $1 - q_0^1 + q_0^2$ we have the sincere strategy set $\mathbf{V}_0^3 = [5, 1, 0]$.

Iteration 1

Given any of the aggregate vote vectors $\mathbf{V}_0^1, \mathbf{V}_0^2, \mathbf{V}_0^3$, each elector individually notices that they neither created nor broke a desirable first-place tie. Since all electors have positive costs but now expect 0 utility from voting, the best response for all electors is to abstain. However, ϕ_6 still has some probability of trembling, so with probability q_1^1 we have $\mathbf{V}_1^1 = [0, 1, 5]$, with probability q_1^2 we have $\mathbf{V}_1^2 = [1, 0, 5]$, and with probability $1 - q_1^1 + q_1^2$ we have the best response strategy set $\mathbf{V}_1^3 = [0, 0, 6]$.

Iteration 2

Given any of the aggregate vote vectors $\mathbf{V}_1^1, \mathbf{V}_1^2, \mathbf{V}_1^3$, each elector individually notices that by voting they could have broken a first-place tie, so with probability q_2^1 we have $\mathbf{V}_2^1 = [4, 1, 1]$, with probability q_2^2 we have $\mathbf{V}_2^2 = [4, 2, 0]$, and with probability $1 - q_2^1 + q_2^2$ we have $\mathbf{V}_2^3 = [5, 1, 0]$.

The aggregate vote total will never repeat with certainty, because ϕ_6 retains their probability of trembling in every iteration, so there can be no certain sequence in the aggregate totals. However, there are 5 electors in the system who will with certainty retain a cycle of period 2; regardless of the choices of ϕ_6, all elecs $\phi_i, i \in \{1; 5\}$ will vote for their most-preferred candidate in any iteration t which satisfies $t \equiv 0 \pmod 2$ and will abstain with certainty in any iteration satisfying $t \equiv 1 \pmod 2$.

Example 5 demonstrates that order can dominate in a system which is not exactly orderly by the aggregate vote totals. In an electorate of size N, we can come up with arbitrarily many examples of situations in which $N - 1$ electors are cycling or stable but one elector is not; such a system can be straightforwardly analyzed, but if we insist on it settling down completely before declaring that it is at or near equilibrium, we will be ignoring deep order and stability within the system. The idea that aggregate totals might have a small amount of noise while the system itself is extremely regular and analysable is exactly the motivation behind Part i) of Definition 5. This discovery immediately generates two related theorems:

Theorem 8. *If there exists any elector in an electorate who maps some aggregate vote choice onto multiple strategies with positive probability, then the aggregate vote total does not cycle.*

Proof. Consider a cycling sequence of aggregate vote totals, so $\exists k \in \mathbb{N}$ so that at equilibrium it always holds that $\mathbf{V}_t = \mathbf{V}_{t+k}$. If the electorate contains some elector who maps an aggregate vote choice onto multiple strategies with positive probability, then there exists some positive probability that the strategy chosen by the elector in iteration $t+1$ as a response to $\mathbf{V}_t$ is not necessarily the same as the strategy chosen in iteration $t+k+1$ in response to the same aggregate vote vector. Therefore, it is not necessarily true that $\mathbf{V}_{t+1} = \mathbf{V}_{t+k+1}$. This is a contradiction with the claim that the electorate's aggregate vote totals are cycling. □

Theorem 9. *There can exist cycling electors in a noncycling electorate.*

Proof. Suppose the negation of this proposition, that there cannot exist cycling electors in a noncycling electorate. Example 5 is a counterexample to this claim. □

The particular reason that Theorem 9 is true is that multiple different vote totals can prompt the same strategy in a given elector.

Next consider the case in which $N = \eta$. It might be tempting to claim that one example of this case is a limit cycle (in continuous mathematics; the finite numerical precision of any machine means that a limit cycle will eventually produce a cycle of exactly period 1 as soon as it produces a number which is equal to the value of the paramter at the rest point $\pm\epsilon_{\text{machine}}$). However, in the case of a limit cycle, Definition 5 Part i) will obtain, so we consider this to be an equilibrium. Where this example occurs is if we have a model in which electors' translation from aggregate vote total to best response strategy includes a random component. Such an example is trivial.

Example 6. *No sequences or stability*

1 elector ϕ_1 choses between candidates ξ_1, ξ_2 and abstention A in each iteration by sampling a value $u_t \sim \mathcal{U}(0;1)$, and voting for ξ_1 iff $0 \leq u_t < \frac{1}{3}$, voting for ξ_2 iff $\frac{1}{3} \leq u_t < \frac{2}{3}$, and abstaining iff $\frac{2}{3} \leq u_t \leq 1$. This system will never cycle or reach a stable state.

Finally, consider the case in which $N = x\sigma + y\kappa + z\eta$. Here we can use a minor modification of Example 5.

Example 7. *Stable, cycling, and neither*

Consider Example 5, but with an elector ϕ_7 with $c_7 > u^{7\to1}$ and $c_7 > u^{7\to2}$. Such an elector always abstains, so the behaviour of the other electors is invariant to its inclusion.

This covers all of the possible types of equilibria that can arise, including an example of each. Using this typology, much more can be said about these equilibrium types; the next section is devoted to proving properties of voting games at equilibrium.

4 Properties of iterative equilibria in voting games

This section is divided into four parts. In Section 4.1, I will prove several theorems which I claim apply to any deterministic voting game as described in Definition 3. In Section 4.2, I will prove several theorems which I claim apply to voting games conducted under the Naïve Pivotality ruleset described in Definition 1. In Section 4.3, I will prove several theorems which apply to the Skellam Pivotality ruleset described in Definition 2. Finally, in Section 4.4, I begin to study how to analyse and plot the set of all equilibria that can arise from one election model.

4.1 Properties of deterministic systems

Let us begin by clarifying what exactly causes different electors to select different strategies.

Theorem 10. *Two electors which have identical preference orderings and identical costs can nevertheless adopt different sequences of strategies.*

Proof. Suppose the negation of this proposition, that no two electors with identical preference orderings and identical costs can ever adopt different sequences of strategies. Proceed by counterexample. First note that trivially these electors can differ in whether or not they abstain in a given iteration, because while their costs are identical, whether or not the cost exceeds the highest strategic utility is directly a function of the magnitudes of their sincere utilities.

I will next prove the stronger assertion that they may vote for different candidates in the same iteration. Consider the simple case of two electors ϕ_0 and ϕ_1 deciding to vote between two candidates ξ_0 and ξ_1 which are involved in a first-place tie or near-tie with another candidate ξ_2, where both electors hold the strict preference ordering $\xi_0 \succ \xi_1 \succ \xi_2$, and setting aside abstention for the moment. Then ϕ_0 votes for ξ_0 over ξ_1 in some iteration t iff

$$p_t(\xi_0, \xi_2)(u^{i\to 0} - u^{i\to 2}) > p_t(\xi_1, \xi_2)(u^{i\to 1} - u^{i\to 2})$$

By the nonnegativity of probabilities, supposing that $p_t(\xi_1, \xi_2) \neq 0$ (which can be motivated by the fact that both of these candidates are assumed to be involved in a first-place tie or near-tie), and assuming nonnegative utilities (the result is identical with negative utilities allowed, but possibly with the inequality reversed) then ϕ_0 will cast a vote for ξ_0 over ξ_1 iff

$$\frac{p_t(\xi_0, \xi_2)}{p_t(\xi_1, \xi_2)} > \frac{u^{i\to 1} - u^{i\to 2}}{u^{i\to 0} - u^{i\to 2}}$$

Define $f \equiv \frac{p_t(\xi_0,\xi_2)}{p_t(\xi_1,\xi_2)}$, then the condition reduces to

$$f > \frac{u^{i\to 1} - u^{i\to 2}}{u^{i\to 0} - u^{i\to 2}}$$

Because nothing has been assumed about electors' sincere utility values, this condition may be satisfied for one elector with a given preference ordering and cost but not satisfied for a different elector with the same preference ordering and cost, which will produce different sincere vote choices. I omit the full counterexample. □

Theorem 10 might be surprising, but it demonstrates how important the precise utility values are in determining what an electors' best response strategies are. However, this finding is not directly useable, because there is no clear way to assign meaning to different utility values across individual electors. Actually, it is conventional to assert that utilities are not comparable between individuals, and the common solution is to normalise the utility values of different electors so that the arbitrary scales of different electors' utility values will not be important in driving results. Normalised utility values are the special case in which there is one number which every elector holds to be their greatest utility value (although they may assign this number to different candidates), their second-greatest utility value is also exactly equal, and so on. This assumption is dramatically simplifying, and will provide crucial leverage over the system, as demonstrated by Theorem 11 and Theorem 12.

Theorem 11. *When utility values are normalised between electors who have identical preference orderings and identical costs, these electors must adopt the same sequences of strategies.*

Proof. In the case of normalised utilities, sincere utilities are entirely determined by preference ordering. Since all electors are identical up to sincere utility values, preference orderings, and costs, if preference orderings and costs are identical between two electors and utility values are normalised, those two electors are exactly identical. Since the system is deterministic, those electors will choose identical sequences of strategies. □

Theorem 12. *When utility values are normalised between electors who have identical preference orderings but different costs, these electors must adopt the same sequences of strategies up to abstention.*

Proof. If the utility values of all electors are normalised, then the inequality in the proof of Theorem 10 can never be satisfied for one elector while also not being satisfied for another elector with an identical preference ordering. So, when these electors vote, they can only vote for the same candidate. However, if two electors have different costs but identical sincere utilities, nothing is directly known about the relative sizes of their costs and their strategic utilities, so one may abstain while the other does not. □

Theorem 11 and Theorem 12 are immensely simplifying statements about the behaviours of different types of electors in any deterministic voting game. Respectively, the situations described by Theorem 10, Theorem 11, and Theorem 12 are modeled by the supplementary files called "all_identical_orderings_and_costs.py", "two_identical_orderings_and_costs_no_abstention.py", and "two_identical_orderings_and_costs_with_abstention.py" on the GitHub code repository for this project.

Next, we will examine some properties of the possible cyclicities of deterministic voting games.

Theorem 13. *If a deterministic voting game is k-cyclic, so that for any iteration t at equilibrium, $\mathbf{V}_t = \mathbf{V}_{t+k} = \mathbf{V}_{t+2k} = \dots$, then $\nexists j$ satisfying $j \not\equiv 0 \pmod{k}$ so that the system is also $j-$cyclic.*

Proof. Suppose that in some k-cyclic deterministic voting game $\exists j \in \mathbb{N}$ satisfying $j \not\equiv 0 \pmod{k}$ so that the system is also $j-$cyclic. Because the system is k-cyclic, $\mathbf{V}_t = \mathbf{V}_{t+k} = \mathbf{V}_{t+2k} = \cdots$, while also $\exists t'$ so $\mathbf{V}_t = \mathbf{V}_{t'}$ and $\mathbf{V}_{t'} = \mathbf{V}_{t'+j} = \mathbf{V}_{t'+2j} = \cdots$. Then by the determinism of the system $\mathbf{V}_t + 1 = \mathbf{V}_{t'} + 1, \mathbf{V}_t + 2 = \mathbf{V}_{t'} + 2, \dots, \mathbf{V}_t + k = \mathbf{V}_{t'} + j$. So $j = k$, which is a contradiction. □

Theorem 13 shows that if there is a k-cycle in the aggregate vote totals of a deterministic voting game, then the totals involved in the cycle exist only in iterations that are multiples of k. This is a useful result for establishing several of the following theorems.

Theorem 14. *In any deterministic voting game, k-cyclic electorates contain only electors whose cycles divide k.*

Proof. Proceed by counterexample. Suppose that the aggregate vote totals in an electorate are k-cyclic, so that for any iteration t at equilibrium, $\mathbf{V}_t = \mathbf{V}_{t+k} = \mathbf{V}_{t+2k} = \cdots$. Also allow that the system contains some ϕ_i which is j-cyclic, so $\forall t$ at equilibrium it picks a sequence of strategies satisfying $s_t = s_{t+j} = s_{t+2j} = \cdots$. Then $\forall t'$ satisfying $t' \equiv 0 \pmod{k}$ and also $t' \not\equiv 0 \pmod{j}$, we have $\mathbf{V}_t = \mathbf{V}_{t'}$, but by Theorem 13, $s_t \neq s_{t'}$. Suppose first that $j|k$. Then there are no t' satisfying this condition, so there is no contradiction in this case. Next suppose that $j \not| k$. Because the aggregate vote count includes elector ϕ_i, this is a contradiction. □

There is a very straightforward interpretation of this result: a system composed of multiple types of cyclers should be understood as k-cyclic, where k is the least common multiple of all of the different cycles that exist in the system. Indeed, what we are really saying when we call a system k-cyclic is that k is the smallest natural number so that all of the electors in the system are k-cyclic. Such a k is of course exactly the least common multiple of all the individual-level cycles in the system.

Theorem 14 is modeled by several models in the GitHub code repository, and when run, three of these models are set to break if this theorem is ever violated, just to further verify the claim. These three models in the repository called "deterministic_diverse_costs.py", "deterministic_epsilon_costs.py", and "skellam_periodicity.py". In 5,031,984 runs of "deterministic_diverse_costs.py", 5,669,437 runs of "deterministic_epsilon_costs.py", and 184,382 runs of "skellam_periodicity.py", Theorem 14 was never once violated. In all of these runs, there was never any k-cyclic electorate which contained an elector of cyclicity j, where $k \neq nj$ for some $n \in \mathbb{N}$. It is however common in some k-cyclic deterministic models to see j-cyclic electors where $j|k$.

This theorem immediately suggests a simplifying theorem:

Theorem 15. *In any deterministic voting game, when k is prime, k-cyclic electorates contain only electors whose shortest cycle has either exactly period k or period* 1.

Proof. This theorem is a direct result of Theorem 14. □

The above theorem dramatically simplifies the procedure of checking the cyclicity of individual electors. In small-M models it is overwhelmingly common for an electorate to have cyclicity of (say) 2, 3, or 5. By Theorem 15, in an electorate of cyclicity 2, every elector must certainly have a cyclicity of 1 or 2. Every single elector in a 7-cyclic electorate has cyclicity exactly 1 or 7. And so on. This means that simply by checking the periodicity of the aggregate vote totals, we know the periodicity (though not necessarily the exact strategies chosen by) every elector in the system. In an electorate of N electors and M candidates that lasts for T iterations at equilibrium, this theorem could save election modeling practitioners equality checks over arbitrarily many combinations of $N \cdot (M+1) \cdot T$ values (with 1 added to M to account for the total abstaining each iteration). Since one might check equality between each value in that list and the analogous value in an adjacent iteration to check for 1-cycling, the value 2 iterations away to check for 2-cycling, and so on, for realistic electorate sizes and party systems that represents arbitrarily many comparisons of $\approx 10^8$ values, in every single run of the iterative model. So aside from being theoretically tidy, Theorem 15 has an immediate practical use.

Another quite simplifying theorem comes straight from the definition of a deterministic voting game:

Theorem 16. *In any deterministic voting game, if the aggregate vote counts remain static from one iteration to the next, then all electors must also remain static.*

Proof. This theorem follows directly from Definition 3. □

Theorem 16 contains the simplifying observation that an elector who is conducting pivotality calculations does not take into account how specific other electors are voting; all that matters is the total number of electors pursuing each strategy.

Theorem 17. *It is not possible to have just one k-cycler in a k-cyclic electorate.*

Proof. Suppose that a k-cyclic system contains exactly one elector ϕ_i with period k, and all other electors have a period less than k. If $k = 1$ then this is trivially a contradiction for any multi-elector system, so it must be that $k > 1$. Then consider some iteration t at equilibrium. All electors other than ϕ_i must pursue the same best response strategies in iteration t as in iteration $t + k'$, where $k' < k$. By the determinism of the system, ϕ_i must select the same best response strategy in all iterations of the form $t + n \cdot k' + 1$ as they did in iteration $t + 1$, $\forall n \in \mathbb{N}$, so the minimum cyclicity of ϕ_i is at most k', not k. This is a contradiction, so no such ϕ_i can exist. □

However, there can exist electors who uniquely have less cyclicity than the system overall:

Theorem 18. *It is possible to have just one k'-cycler in a k-cyclic electorate with $k' < k$.*

Proof. Suppose that no k'-cycler can exist alone in a k-cyclic electorate with $k' < k$. Then consider the counterexample of a 2-cyclic elector that switches back and forth between abstention and their sincere vote choice in an electorate which otherwise only contains 4-cyclic electors. Trivially such an elector can exist, for example if the vote totals of their sincerely preferred candidate is higher in all odd iterations than in all even iterations. □

The case established in Theorem 18 turns out to be a common result in many of the supplementary models that were written for this project.

4.2 Properties of Naïve Pivotality

The following theorems hold for voting games which employ the Naïve Pivotality ruleset described in Definition 1.

Theorem 19. *In any voting game which uses Naïve Pivotality, k-cycling is necessarily the result of $k - 1$ consecutive ties or ties ± 1.*

Proof. Suppose that some sequence of aggregate vote totals exhibits a k-cycle, so that for any iteration t at equilibrium, $\mathbf{V}_t = \mathbf{V}_{t+k} = \mathbf{V}_{t+2k} + \ldots$. Consider iteration t: either there is a tie or a tie ± 1, or neither is present. Notice that it must be the case that there exists at least one elector ϕ_i with $c_i > 0$, because if no such elector exists then all electors will vote sincerely in every iteration, which is a contradiction with the assumption that the aggregate vote totals cycle with period k. With this observation, then if neither a tie nor a tie ± 1 exists in iteration t, the set of aggregate vote counts in $t+1$ is necessarily different from the set of aggregate vote counts in t, because at least one elector, ϕ_i, will abstain in t. In all subsequent iterations, if there is neither a tie nor a tie ± 1, then every elector votes sincerely, while if there is a tie or tie ± 1, then the set of aggregate vote totals changes up to iteration $t+k$, and the sequence repeats. □

Theorem 20. *In any voting game which uses Naïve Pivotality, the vote total v_j^{t+1} of some candidate ξ_j in iteartion $t+1$ is at least as large as that candidate's previous vote total v_j^t if and only if ξ_j was involved in a tie or a tie ± 1 in iteration t.*

Proof. First suppose that $v_j^{t+1} \geq v_j^t$. Then, there must exist at least one elector ϕ_i that did not vote for j in t but does vote for j in $t+1$. Under the naïve pivotality rule, electors vote only for either their sincere preference or for a candidate that was involved in a tie. So if a candidate's vote total grows from one iteration to the next, then they must have been involved in a tie.

Next suppose that the a candidate j was involved in a tie in some iteration t. Begin by considering the set of electors which voted for j in t. By definition these electors prefer j to the other candidate or candidates involved in the tie in t. By the if condition proved above, no candidates can be involved in a tie or tie ± 1 in $t+1$ which were not involved in the tie or tie ± 1 in t. Therefore no elector will switch their vote away from j between t and $t+1$. However, electors which did not support j in t may support j in $t+1$, for example if $v_j^t > v_j^{t-1}$. So if j was involved in a tie, then necessarily $v_j^{t+1} \geq v_j^t$. □

Theorem 21. *The only rest point of the Naïve Pivotality logic is the wasted vote result derived in Cox (1994), in which electors pool onto candidates in either ties or ties ± 1*

Proof. By Theorem 20, the unique rest point of any deterministic voting game is the case in which a set of candidates are involved in a succession of ties or ties ± 1 in which no additional electors support the candidates from one iteration to the next. □

Theorem 21 allows us to understand the Cox (1994) wasted vote result as the rest point of an iterative dynamical system. In any system in which electors vote only when they expect to create or break a first-place tie, there is one unique situation in which every elector will vote with certainty: this is the situation in which there is certainly a tie. In every other situation, there will be infinite cycling. Theorem 21 motivates why the Naïve Pivotality logic might be of interest; this represents a replication of the Cox (1994) result in an iterative voting framework.

Theorem 22. *In any voting game which uses Naïve Pivotality, k-cyclic electorates contain only electors who cycle with period exactly k or 1.*

Proof. Consider an electorate with M candidates that produces vote counts which repeat with some finite period. By Theorem 14, if $\exists \phi_i$ so that the shortest cycle of ϕ_i does not equal the period of the aggregate vote counts, then the period of the cycle in the strategies of ϕ_i must divide the period of the aggregate vote counts. So allow that there are $y \in \mathbb{N}$ groups of electors with periods that divide the period of the electorate. Now consider the first iteration which is at equilibrium, call it t. Say that the least cycle which is present in the electorate has period k; this is the smallest number of iterations that it takes for any elector in the population to repeat with certainty.

Fix on one elector who cycles with period k. In t say that this elector adopts strategy s_1, followed by s_2 in $t+1$, and so forth, repeating at $t+k$. First consider the case in which the elector never chooses to abstain. Then each strategy in the cycle corresponds to a preference of the corresponding candidate over the other candidates involved in the tie. This means that in iteration t the elector prefers candidate 1 over any other candidate involved in that tie or near-tie, in iteration $t+1$ the elector prefers candidate 2 over any other candidate involved in that tie or near-tie, and so on. This means that the tie in iteration t includes candidate 1, the tie in iteration 2 includes candidate 2 but does not include candidate 1, the tie in iteration 3 includes candidate 3 but not candidates 1 or 2, and so on. This is because, from the fact that candidates 1 and 2 were both involved in the first-place tie in iteration t and the elector chose to vote for candidate 1, we can infer that for this elector, candidate 1 is preferred to candidate 2, candidate 2 is preferred to candidate 3, and so on. But a candidate's vote total can increase from one iteration to the next only if it is involved in a first-place tie or near-tie in the previous iteration; otherwise it can receive at most the same number of votes that it received in the previous iteration. Likewise, all candidates are preferred to the candidate one after them. Now, notice that in iteration $t+k-1$, the elector votes for candidate k, but it also follows that candidate 1 can only be present in the first-place tie in iteration $t+k+1$ if it was present in the first-place tie in iteration $t+k-1$. This means that the elector has the strict preference ordering $1 \succ 2 \succ \cdots \succ k \succ 1$. This violates the transitivity of the individual electors. Even relaxing transitivity, this also contradicts the requirement that candidate 1 was not involved in that tie.

Next, relax the assumption that abstention is not permitted. So consider an elector with the strategy sequence $\{s_1, \ldots, s_k\}$, where the strategies can be abstention, and the elector is part of a larger repeating electorate. ϕ_i picks $\{s_1, \ldots, s_k\}$, where the strategies may be abstention. ϕ_i abstains iff no tie or tie ± 1 exists from which they would obtain a higher payoff than their voting cost. So abstention in iteration t means iteration t features a tie between candidates $\xi_t^1, \ldots, \xi_t^w$ where $\max(u^{i \to j}) < c_i \ \ \forall j \in \{1, \ldots, w\}$. Because ϕ_i cannot only abstain (which would be a 1-cycle), and because $s_k \to s_1$, some abstention must be followed by a vote for a candidate. So some strategy with abstention precedes a strategy where $\max(u^{i \to j}) > c_i$ for some $j \in \{1, \ldots, w\}$. Because vote totals only increase through ties or near ties, this is a contradiction.

Finally, relax the assumption that the elector never repeats strategies within a k-cycle. Then, it must be the case that $\exists h \in \{1, \ldots, k\}$ with $h < k$ so that s_h is played more than once in the sequence $\{s_1, \ldots, s_k\}$. Then we can infer that $1 \succ 2 \succ \cdots \succ h \succ \cdots \succ h \succ \cdots \succ k$. Transitivity and completeness cannot both hold.

This covers all the cases, so in a $k-$cyclic electorate with Naïve Pivotality updates, all electors must have period exactly k. □

In 10,701,421 runs of models that used the Naïve Pivotality updating rule, which included about 100,000,000 electors, no single elector ever cycled with a different period than the period of the aggregate vote totals.

4.3 Properties of Skellam pivotality

The following theorems hold for voting games which employ the Skellam pivotality ruleset described in Definition 1.

Theorem 23. *Without abstention, the iterative Skellam model always hits the deterministic rest point, which is a first-place tie or tie* ± 1.

Proof. This result follows a direct analog of Theorem 21. □

Theorem 24. *The skellam model is always* k*-cyclic, with* $k \geq 1$.

Proof. Theorem 24 result follows from Theorem 7, because once the population size is randomly sampled, the model is deterministic. □

Theorem 25. *Voting games which employ Skellam Pivotality can have* k'*-cyclers in a* k*-cyclic model, where* $k'|k$ *and* $k' \neq k$.

My proof for Theorem 25 is a very involved counterexample, and does not provide any real insight into why the result is true. I omit the full proof, and instead discuss the intuition.

The core reason that every k-cyclic Naïve Pivotality voting game can contain only $k-$cyclic or stable electors was that the voting rule is so simple that every elector essentially acts in the same way in each iteration. If there is a reason to vote sincerely to break a tie, every elector will jump onto that tie; if there is no such tie, then every elector will either abstain or vote sincerely, depending on their cost. Using Skellam Pivotality logic introduces a wrinkle: for certain sets of electors' cost values and preference orderings over candidates', there can arise iterations where the vote totals of the leading candidate and the second-place candidate are distant enough for many electors to abstain, but one or more electors to continue casting a strategic vote.

Remark 26. *Switching from one candidate to another is extremely rare. Switching from abstention to one specific favoured candidate and then back to abstention is much more common.*

Of the 184,382 runs in that I checked for the behaviours described in Remark 26, which included about 1,500,000 electors, only 7,506 runs included at least one elector who voted for more than one candidate after the model hit iterative equilibrium. This is true even though a majority of runs had aggregate cyclicities greater than 1. These results are from the script called "skellam_periodicity.py"in the GitHub repository.

4.4 Searching possible equilibria

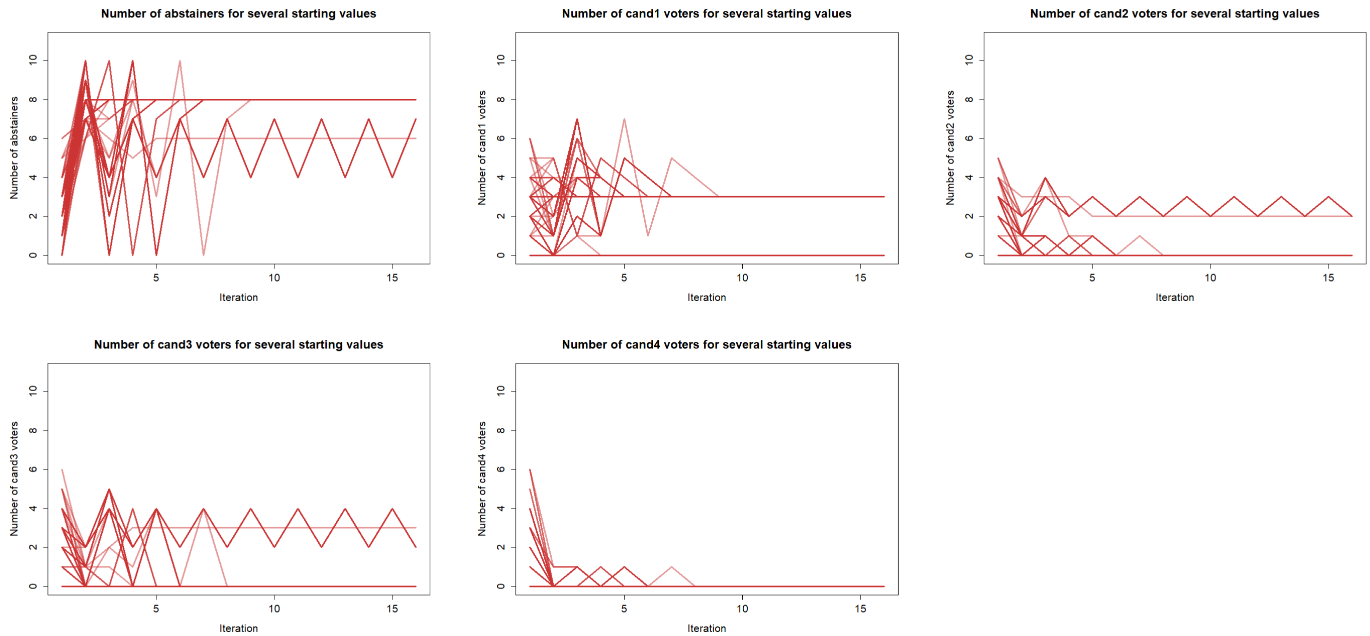
The central concern of this paper was to define a notion of equilibrium that is usable for iterative voting games, and to investigate what behaviours are possible at equilibrium, with a particular focus on individual-level strategies. I have however said comparatively little about what the equilibria themselves look like. Because it is a core strength of computational modeling paradigms that they can vary the starting conditions of a model so as to cover the full space of possible equilibria (Bendor et al., 2011, Siegel, 2018), I end this paper with a brief subsection in which I directly examine the space of possible equilibria in some example iterative voting games. I will begin by describing an idea for what part of the system to vary to usefully turn up multiple possible equilibria, then I will show results from a novel software tool that sweeps over initial states and plots the resultant equilibria, and finally explicitly connect the discussion of iterative equilibria to dynamical systems notions of equilibria.

Consider the simplest dynamical systems interpretation of what it means to consider two different equilibria in the same system: here we should vary absolutely no details about the system except the "seed", the set of initial values, which in this case is the set of initial votes that electors cast. Rather than having electors begin by voting sincerely, imagine that electors instead cast their initial votes randomly, so that the system begins at some point in an $M + 1$-dimensional space (one value specifying the number of votes for each candidate, and abstention). If we begin by randomly selecting a variety of different seeds for the system, then we can check how many different equilibria it is possible to obtain simply by changing the first vote values that electors see.

In Figure 1, I show the possible vote counts that can arise from a variety of different initial vote counts in the same Skellam Pivotality model. In these runs, I begin by instantiating a set number of electors with preference orderings over a certain number of candidates. I then randomize the initial vote counts, and observe what the equilibrium of the system is, without varying the number of electors, their preferences, or the number of candidates. Each subplot shows the support for a candidate or abstention over time, and each line in these plots shows the progression of one run of the model given a particular randomly drawn initial vote count,[10] and the darkness of the lines is proportional to how many runs of the model produced that value. In the particular model that I show in Figure 1, I identified three distinct equilibria in the 100 sweeps of the model that are shown in that figure.

[10] I randomly draw initial vote counts rather than stepping over the space and covering all of it because the combinatorial options for initial votes is truly explosive for non-trivial electorate sizes.

Figure 1: Evolution of different initial vote counts



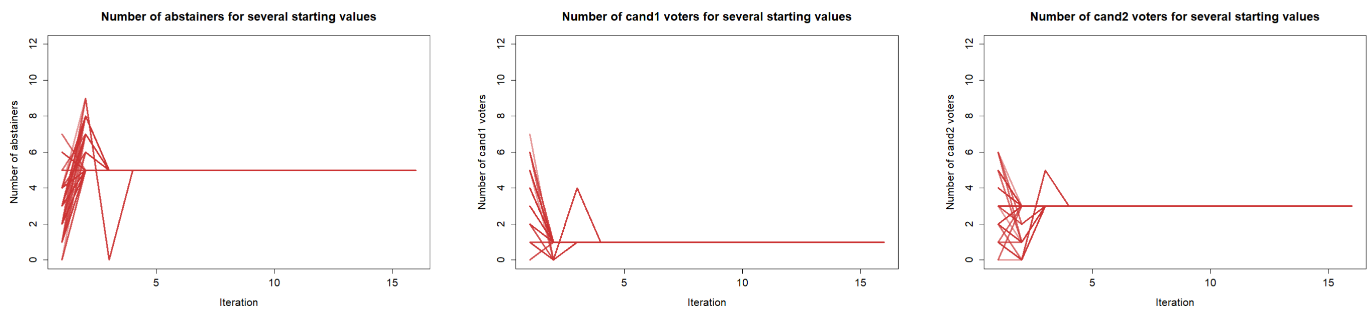
Out of the three equilibria identified by this method, two were stable. One equilibrium, which featured 8 abstainers and 3 electors voting for (the candidate which the model automatically labeled) candidate 0, accounted for a full 82% of the runs. The other stable equilibrium, in which 6 electors abstained, 2 voted for candidate 2, and 3 voted for candidate 3, only appeared in 1% of the runs. And the unique cycling equilibrium, in which electors vacillated between a state in which 7 electors abstained with 2 supporting candidate 1 and 2 supporting candidate 2 and a state in which 4 electors abstained with 3 supporting candidate 1 and 4 supporting candidate 2, comprised the remaining 17% of the runs. It's interesting to note that all of the cycles were perfectly synchronized; whenever the system cycled at equilibrium, odd iterations always had 7 abstainers and even iterations always had 4 abstainers.

I present this information to demonstrate two important methodological points. First, we need a method to sweep over possible equilibria and visualise them, or else we will miss the incredibly rare equilibria that can appear. This thoroughly average simulation run had an equilibrium (the stable one in which 6 electors abtained) which only appeared in 1% of the runs, so in order to expect to see it once I needed to run the model 100 times. To thoroughly understand even very simple iterative voting games, we need to have a flexible framework that explores the space of possible starting values at a sufficiently large scale to catch the rare equilibria. Notice that none of these equilibria were substantively ignorable; in 82% of the runs candidate 0 would win, in 1% of the runs candidate 3 would win, and in half of the iterations of 17% of the runs candidate 2 would win, while in the other half of the iterations of those 17% of the runs there would be a tie between candidate 1 and candidate 2. Every candidate was at least tied for first in some equilibrium.

The second purpose of this discussion has been to demonstrate that I have begun to develop the framework necessary to get some capacity to analyse the multiplicity of equilibrium states in iterative voting games. The start of that software suite can be found among my project code on GitHub, specifically in the files “equilibrium_plotter.r”, “deterministic_equilibrium_searching.py”, and “skellam_equilibrium_searching.py”.

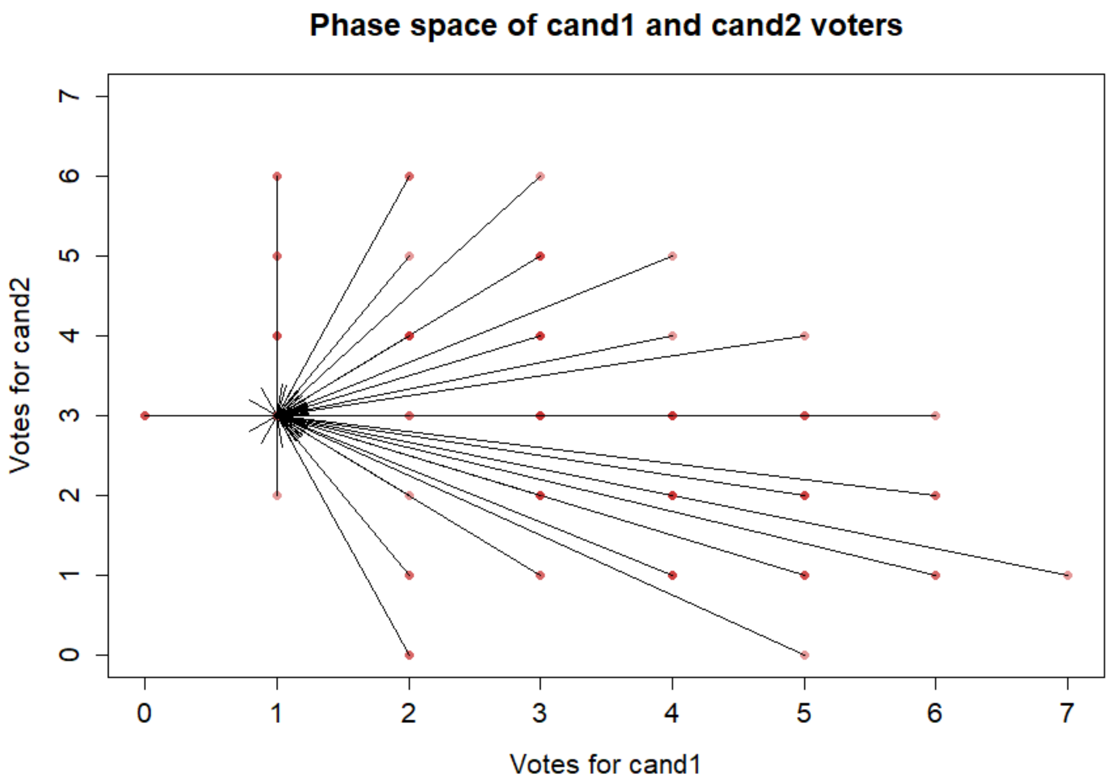
With the capacity to plot equilibria, we can visualise the stability of various equilibria, which immediately connects our notion of equilibria to how equilibria are conceived of in dynamical systems theory. It is evident from looking at Figure 2 that the voting game it depicts has an extremely strong sink; no matter what the starting values of the system are, 100% of the runs collapse almost immediately to the situation in which 5 electors abstain, 1 votes for candidate 1, and 3 vote for candidate 2.

Figure 2: Stability of different initial vote counts



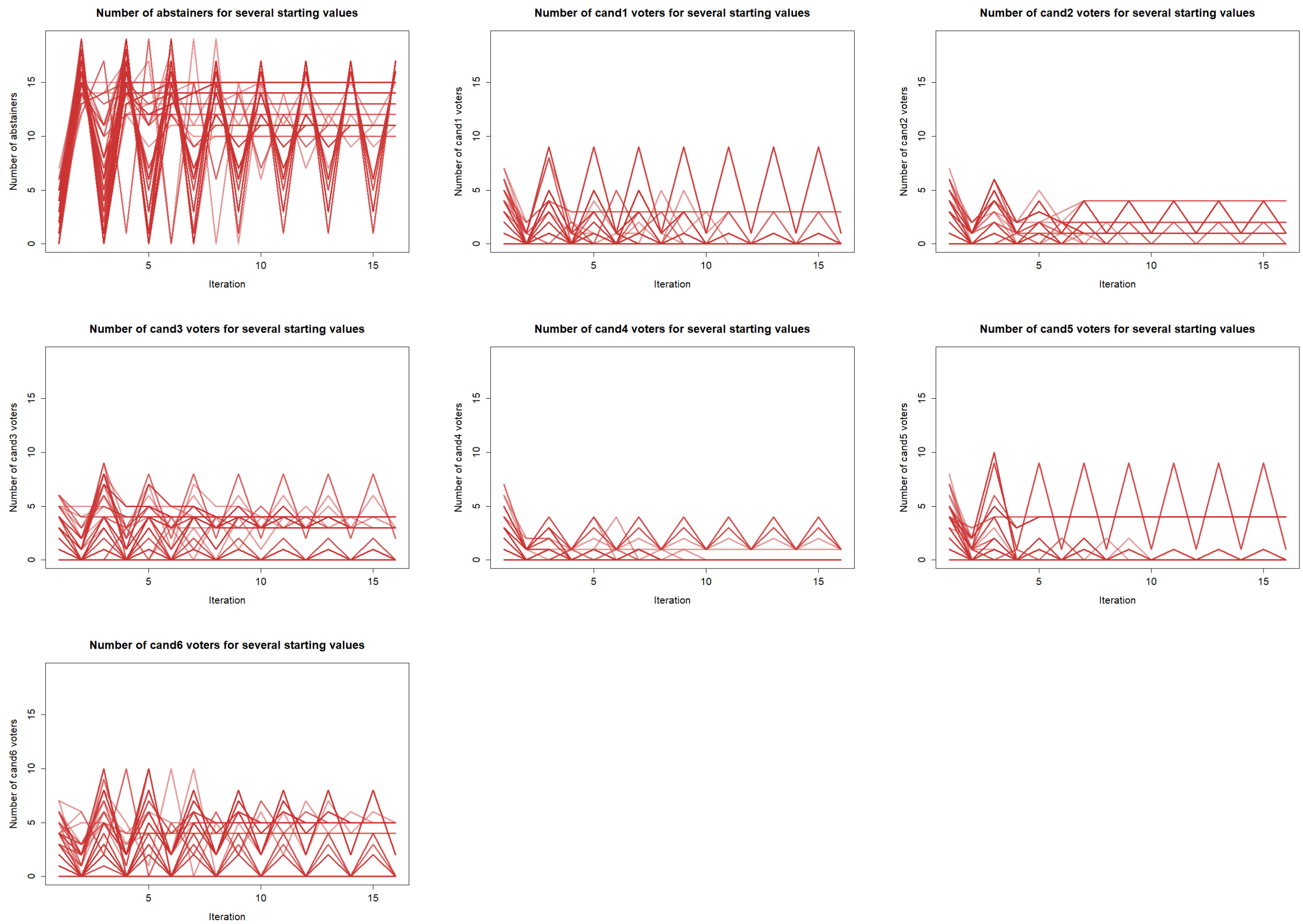
This immediately suggests plotting the system as a phase space, as is conventional for displaying dynamical equilibria (Strogatz, 1994). We can naïvely do that in the case of 2 candidates by plotting the phase space of one candidate’s vote total against the other candidate’s vote total, since this is a 2 dimensional space (setting aside abstention for the moment). We can very crudely represent time by simply including arrows which point from the starting value of the system in one run to the ending value of the system in the same run. If we plot that for the same system shown in 2, it becomes exceptionally clear that there is a sink which is the inevitable end state of a model which begins anywhere in the entire explored system.

Figure 3: Stability of different initial vote counts



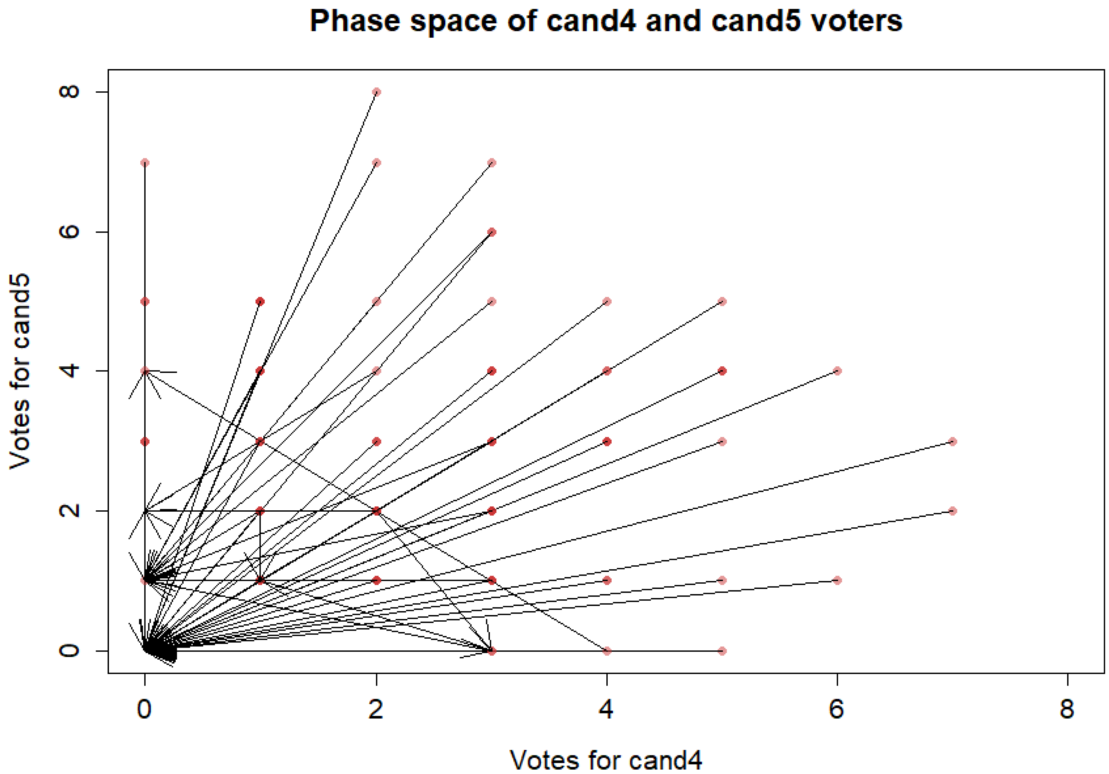
Finally, if we wish to see more variety in the types of equilibrium, there is a very clear parameter to increase: the number of candidates. The number of possible equilibria increases dramatically in the number of candidates. Figure 4 shows the multiplicity of equilibria in a 6-candidate election.

Figure 4: Evolution of different initial vote counts



To represent the full phase space would require a 7-dimensional image, but even just plotting the phase space between any two candidates of interest demonstrates how many more possible equilibria there are in this system than in the 2-candidate system. This phase portrait is shown in Figure 5.

Figure 5: Stability of different initial vote counts



The complex phase portrait in Figure 5 presents a strong contrast to the one simple sink in Figure 2. The equilibria in the 2-dimensional cross-section of a 7-dimensional system looks like it should evade any sort of easy description, even though this system contains only 19 electors and 6 candidates. This further illustrates why the simplifying theorems throughout the past several sections promise to be useful; there is actually immense order in the behaviour of the electors, concealed underneath this complex many-dimensional phase portrait. Nevertheless, developing more sophisticated routines for plotting equilibria in iterative voting games is one of the very highest priorities for future work in this project.

There is one deep complication with the preceding discussion: varying the seed of the system violates the setup of the game. There is no reason for electors to ever randomly vote, or to expect other electors to randomly vote. I propose two directions for future work in this regard. The first is that, by modeling repeated elections, we could construct lists of possible starting states under the actual rules of the game. These several starting states may or may not produce multiple equilibria. This is a middle ground between randomly searching the entire space and only picking one arbitrary seed, and as a middle ground it has some promise to actually capture the chaotic dynamics of real elections: seemingly spontaneous events do occur with some frequency, but they also exhibit stability over time in many parameters (Saari, 2001). The second direction, which is related, is to flesh out candidates to be more than just the empty objects of electors' preferences that they are in my model, so that deeper facets of the system can be randomly varied without effectively creating a completely new model. For example, if candidates were assigned to values in a space and electors' ideal points were varied, that might be a more theoretically grounded means of seeding the system randomly.

5 Conclusion

This paper defined an equilibrium notion that reduces the gap between the results of iterative computational formal models and their analytical game theoretic analogs. I began by precisely defining the system of interest, and I used this system to illustrate my novel definition of iterative equilibria in computational models of political processes. Using voting games to illustrate this idea, I catalogued the possible end-states of electors, and provided detailed examples of how each could exist in a hypothetical iterative voting model. I then stated and proved 26 theorems which mostly concerned the behaviours of iterative voting games at equilibrium.

In this paper I showed that every deterministic iterative voting game is cyclical with a known upper bound, and then proved a series of strong relationships between the patterns in aggregate- and individual-level parameters in these cyclical equilibria. I also proved useful limitations on electors' behaviours in the system, and showed that both pivotality updating rules result in the $M + 1$ result in Cox (1994) if and only if they reach a rest point. I then showed preliminary work on the open problem of trying to study all of the possible equilibria in an iterative voting game. The 26 findings that I prove in this paper were double checked and illustrated using more than 11,000,000 runs of eight computer simulations that I wrote, which include some software which can immediately be used in much larger computational models of elections. The generality of my definitions and the strength of the regularities I find in iterative election games suggest that my notion of iterative equilibrium can be applied to political processes other than just elections.

References

Acemoglu, D., Egorov, G., and Sonin, K. (2009). Equilibrium refinement in dynamic voting games. *Paper presented at the Massachusetts Institute of Technology Department of Economics Working Paper Series*.

Adami, C. (1998). *Introduction to artificial life*. Springer-Verlag.

Arnol'd, V. I. (1973). *Ordinary Differential Equations*. MIT Press, 1 edition.

Axelrod, R. (1981). *Evolution of cooperation*. Basic Books.

Axelrod, R. (1997). *The complexity of cooperation: Agent-based models of competition and collaboration*. Princeton University Press.

Azbelev, N., Maksimov, V., and Rakhmatullina, L. (1995). *Introduction to the theory of linear functional differential equations*. World Federation Publishers Company.

Bainov, D. D. and Covachev, V. (1994). *Impulsive differential equations with a small parameter*. World Scientific Publishing, 1 edition.

Bainov, D. D. and Simeonov, P. S. (1995). *Impulsive differential equations: Asymptotic properties of the solutions*. World Scientific Publishing, 1 edition.

Baker, G. L. and Golub, J. P. (1996). *Chaotic dynamics: An introduction*. Dover Publications, 2 edition.

Baltz, S., McAlister, K., Pineda, A., Vasselai, F., and Walter R. Mebane, J. (2018). Abstention in elections with asymmetric information and diverse preferences. *Presented at the 2018 Annual Meeting of the Midwest Political Science Association, Chicago, IL.*

Bellman, R. (1966). *Perturbation techniques in mathematics, engineering & physics*. Dover Publications.

Bendor, J., Diermeier, D., Siegel, D. A., and Ting, M. M. (2011). *A behavioral theory of elections*. Princeton University Press.

Binmore, K. (1990). *Essays on the foundations of game theory*. Basil Blackwell, 1 edition.

Binmore, K. (2009). *Rational decisions*. Princeton University Press, 1 edition.

Booker, L., Forrest, S., Mitchell, M., and Riolo, R. (2005). *Perspectives on adaptation in natural and artificial systems*. Oxford University Press.

Borrelli, R. L. and Coleman, C. S. (1998). *Differential equations: A modeling perspective*. John Wiley.

Bratley, P., Fox, B. L., and Schrage, L. E. (1987). *A guide to simulation*. Springer, 2 edition.

Cox, G. W. (1994). Strategic voting equilibria under the single nontransferable vote. *American Political Science Review*, 88:608–621.

de Condorcet, M. (1785). *Essai sur l'application de l'analyse à la probabilité des décisions rendue à la pluralité des voix*. L'imprimerie Royale, Paris.

Dekel, E. and Scotchmer, S. (1992). On the evolution of optimizing behavior. *Journal of Economic Theory*, 57(2):392–406.

Demichelis, S. and Dhillon, A. (2010). Learning in elections and voter turnout. *Journal of Public Economic Theory*, 12(5):871–896.

Downs, A. (1957). *An Economic Theory of Democracy*. Harper Collins.

Eubank, S. G. and Farmer, J. D. (2003). Introduction to dynamical systems. In Lam, L., editor, *Introduction to nonlinear physics*, chapter 5, pages 55–105. Springer, 1 edition.

Feddersen, T. J. and Pesendorfer, W. (1999). Abstention in elections with asymmetric information and diverse preferences. *American Political Science Review*, 93(2):381–398.

Fey, M. (1997). Stability and coordination in duverger's law: A formal model of preelection polls and strategic voting. *American Political Science Review*, 91(1):135–148.

Fudenberg, D. and Levine, D. K. (1999). *The theory of learning in games*. MIT Press, 2 edition.

Germeier, Y. B. (1986). *Non-antagonistic games*. D. Reidel Publishing Company, 2 edition.

Giaimo, C. (2019). Wasps passed this logic test. can you? *The New York Times*.

Gibbons, R. (1992). *Game theory for applied economists*. Princeton University Press, 1 edition.

Gilbert, N. (2008). *Agent-based models*. SAGE, 1 edition.

Golder, M. (2003). An evolutionary approach to party system stability. *Paper presented at the Annual Meeting of the Midwest Political Science Association, Chicago*.

Grousouzakou, E. and Contopoulos, G. (1998). Distribution of periodic orbits in 2-d dynamical systems. In Benest, D. and Froeschlé, C., editors, *Analysis and modeling of discrete dynamical systems*, chapter 4, pages 107–131. Gordon and Breach Science Publishers, 1 edition.

Hirsch, M. W., Smale, S., and Devaney, R. (2004). *Differential equations and dynamical systems: An introduction to chaos*. Elsevier, 1 edition.

Kim, S., Taber, C. S., and Lodge, M. (2010). A computational model of the citizen as motivated reasoner: Modeling the dynamics of the 2000 presidential election. *Political Behavior*, 32(1):1–28.

Kollman, K., Miller, J. H., and Page, S. E. (1992). Adaptive parties in spatial elections. *American Political Science Review*, 86(4):929–937.

Kollman, K., Miller, J. H., and Page, S. E. (1997). Landscape formation in a spatial voting model. *Economics letters*, 55(1):121–130.

Kollman, K., Miller, J. H., and Page, S. E. (1998). Political parties and electoral landscapes. *British Journal of Political Science*, 28(1):139–158.

Kolmanovskii, V. and Myshkis, A. (1992). *Applied theory of functional differential equations*. Kluwer Academic Publishers.

Konishi, H. and Ray, D. (2003). Coalition formation as a dynamic process. *Journal of Economic Theory*, 110:1–41.

Kuczma, M., Choczewski, B., and Ger, R. (1990). *Iterative Functional Equations*. Cambridge University Press.

Laver, M. and Sergenti, E. (2012). *Party competition: An agent-based model*. Princeton University Press, 1 edition.

Linzer, D. and Honaker, J. (2003). A theory of the evolution of voting: Turnout dynamics and the importance of groups. *Presented at the American Political Science Association Annual Meeting*.

Marotto, F. R. (2006). *Introduction to mathematical modeling using discrete dynamical systems*. Thompson Brooks/Cole.

Martelli, M. (1990). *Introduction to discrete dynamical systems and chaos*. John Wiley & Sons.

McKelvey, R. (1976). Intransitivities in multidimensional voting models and some implications for agenda control. *Journal of Economic Theory*, 12(3):472–482.

Montoro-Pons, J. D. and Puchades-Navarro, M. (2001). An evolutionary model of voting. *European Journal of Economic and Social Systems*, 15(3):171–184.

Myerson, R. B. (2000). Large poisson games. *Journal of Economic Theory*, 94(1):7–45.

Nowak, M. A. and Highfield, R. (2011). *SuperCooperators: Altruism, evolution, and why we need each other to succeed*. Free Press.

Palfrey, T. R. and Rosenthal, H. (1985). Voter participation and strategic uncertainty. *American Political Science Review*, 79(1):62–78.

Paxton, B., Bildsten, L., Dotter, A., Herwig, F., Lesaffre, P., and Timmes, F. (2010). Modules for experiments in stellar astrophysics. *The astrophysical journal*, 192(1).

Perko, L. (2001). *Differential equations and dynamical systems*. Springer, 3 edition.

Quinn, K. M. and Martin, A. D. (2002). An integrated computational model ofmultiparty electoral competition. *Statistical Science*, 17(4):405–419.

Railsback, S. F. and Grimm, V. (2012). *Agent-based and individual-based modeling: A practical introduction*. Princeton University Press, 1 edition.

Rasband, S. N. (1990). *Chaotic dynamics of nonlinear systems*. Dover Publications.

Riker, W. H. and Ordeshook, P. (1968). A theory of the calculus of voting. *American Political Science Review*, 62(1):25–42.

Saari, D. G. (2001). *Chaotic elections! A mathematician looks at voting*. American Mathematical Society.

Sadiraj, V., Tuinstra, J., and van Winden, F. (2004). A computational electoral competition model with social clustering and endogenous interest groups as information brokers. *Working paper*.

Sadiraj, V., Tuinstra, J., and van Winden, F. (2006). Voting cycles in a computational electoral competition model with endogenous interestgroups. *Working paper*.

Samoilenko, A. M. and Perestyuk, N. A. (1995). *Impulsive differential equations*. World Scientific Publishing, 1 edition.

Sandefur, J. T. (1990). *Discrete dynamical systems: Theory and applications*. Clarendon Press.

Schofield, N. (1978). Instability of simple dynamic games. *The Review of Economic Studies*, 45(3):575–594.

Schroeder, M. (1991). *Fractals, chaos, power laws: Minutes from an infinite paradise*. Dover Publications.

Seydel, R. (1988). *From equilibrium to chaos: Practical bufirucation and stability analysis*. Elsevier.

Sieg, G. and Schulz, C. (1995). Evolutionary dynamics in the voting game. *Public Choice*, 85(1,2):157–172.

Siegel, D. A. (2018). Analyzing computational models. *American Journal of Political Science*, 62(3):745–759.

Simon, H. (1983). *Reason in human affairs*. Stanford University Press, 1 edition.

Skellam, J. G. (1946). The frequency distribution of the difference between two poisson variates belonging to different populations. *Journal of the Royal Statistical Society Series A (General)*, 109(3):296.

Smith, A. (2004). *Election timing*. Cambridge University Press.

Smith, J. M. (1993). *The theory of evolution*. Cambridge University Press, Canto, 4 edition.

Smith, J. M. (1998a). *Evolution and the theory of games*. Cambridge University Press, 7 edition.

Smith, J. M. and Price, G. R. (1973). The logic of animal conflict. *Nature*, 246(1):15–18.

Smith, P. (1998b). *Explaining Chaos*. Cambridge University Press.

Strogatz, S. H. (1994). *Nonlinear dynamics and chaos*. Perseus Books Publishing, 1 edition.

Tenenbaum, M. and Pollard, H. (1963). *Ordinary differential equations*. Dover Publications.

Verhulst, F. (2000). *Nonlinear differential equations and dynamical systems*. Universitext, 2 edition.

von Neumann, J. and Morgenstern, O. (1953). *Theory of games and economic behavior*. Princeton University Press, 3 edition.

Watson, J. (2013). *Strategy: An introduction to game theory*. W. W. Norton & Company, 3 edition.

Weibull, J. (1998). *Evolutionary game theory*. MIT Press, 4 edition.

Weisbuch, G. (1991). *Complex systems dynamics*. Addison-Wesley.

Wolfram, S. (2002). *A new kind of science*. Wolfram Media.

Zeigler, B. P. (1976). *Theory of modelling and simulation*. John Wiley & Sons, 1 edition.

6 Glossary of notation

ϕ: An elector
N: The number of electors in an electorate
ξ: A candidate
M: The number of candidates in an election
σ: The number of electors who are static at equilibrium
κ: The number of electors who cycle at equilibrium
η: The number of electors who never reach equilibrium
ε: An arbitrarily small positive real number
ρ: A parameter of a computational formal model
$\mathcal{M}$: A computational formal model
k: The period, also called the cyclicity, of a cycle
x, y, z: Real numbers in $[0; 1]$ to represent proportions of an electorate
s: A strategy
S: A strategy set held by an elector
t: A natural number representing some iteration of a model
T: A total number of iterations
τ: A threshold iteration, usually denoting the first equilibrium iteration
n: An arbitrary natural number
u: A utility value
$u_t^{i \to j}$: The utility that elector i obtains from candidate j in iteration t. No subscript if sincere
$\mathbf{U}$: A matrix of utility values
$\mathbf{E}$: A matrix of expected utility values
c: A cost of voting
$\mathbf{e}$: A vector containing a 1 corresponding to the elector's chosen strategy and a 0 elsewhere
v: The total number of votes for a particular candidate
$\mathbf{V}$: A vector of the total votes for each candidate
A: Abstention
p: A pivotal probability
$\mathbf{P}$: A matrix of pivotal probabilities
g: The probability of creating a tie between any two candidates
h: The probability of breaking a tie between any two candidates
i: Usually denotes some elector in the set of electors
j: Usually denotes some candidate in the set of candidates
Superscripts are usually reserved for agents
Subscripts are usually reserved for iterations