Iterative voting games

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Abstract

Iterative computer models enable the study of repeated processes throughout an election. Because we are freed from the concern of pencil-and-paper solvability, we can now model the behaviour of large and diverse electorates. However, iterative models present their own challenge: how do we define convergence in iterative election models? Using the example of electors calculating their pivotal probability by consulting a sequence of election polls, I demonstrate that a broad category of iterative election models produce cyclic behaviours in the electors' strategic intentions. These methods demonstrate how we can use formal models to study the process of electors consulting polls during a campaign.

The idea

Iterative election game:

↪ Generate a population of electors with preferences over alternatives
↪ Seed the system arbitrarily, say with sincere voting
↪ Every iteration, all electors see a "poll" or census
↪ Electors calculate expected pivotal probability using that aggregated information
↪ Electors calculate their expected utility from each alternative
↪ Electors signal the best response among their strategies
↪ Repeat

Methodology: This scenario arises naturally in computer models of elections (Bendor et al. 2011; Laver and Sergenti 2012; Mebane et al. 2019; Seigel 2018), but has unstudied properties that are absent from these models' game theoretic origins.

Applications:

↪ Electors consulting surveys throughout an election campaign
↪ This is a specific example of a multi-round voting problem
↪ Relaxing simultaneity gives sequential voting ideas like legislative committees

The problem: bouncing equilibria

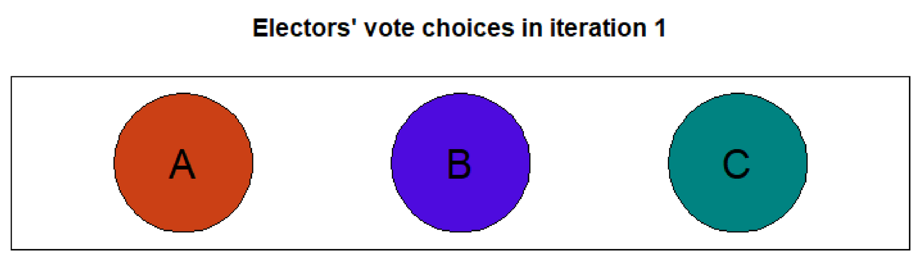
What if an iterative model does not converge?

Example: The preferences in Condorcet's Paradox suggest a nonconvergent iterative game. Preference orderings in Condorcet's Paradox:

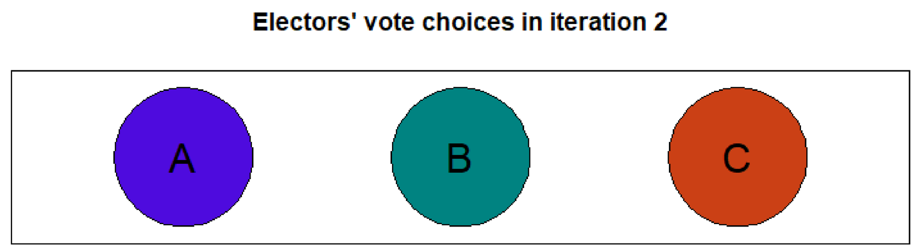
↪ Elector A: RED ≻ BLUE ≻ GREEN
↪ Elector B: BLUE ≻ GREEN ≻ RED
↪ Elector C: GREEN ≻ RED ≻ BLUE

A simple "Compromise" Ruleset: If by switching my vote I could break a first-place tie for a candidate other than my least-preferred candidate, assuming that no other elector will switch their vote, then I will.

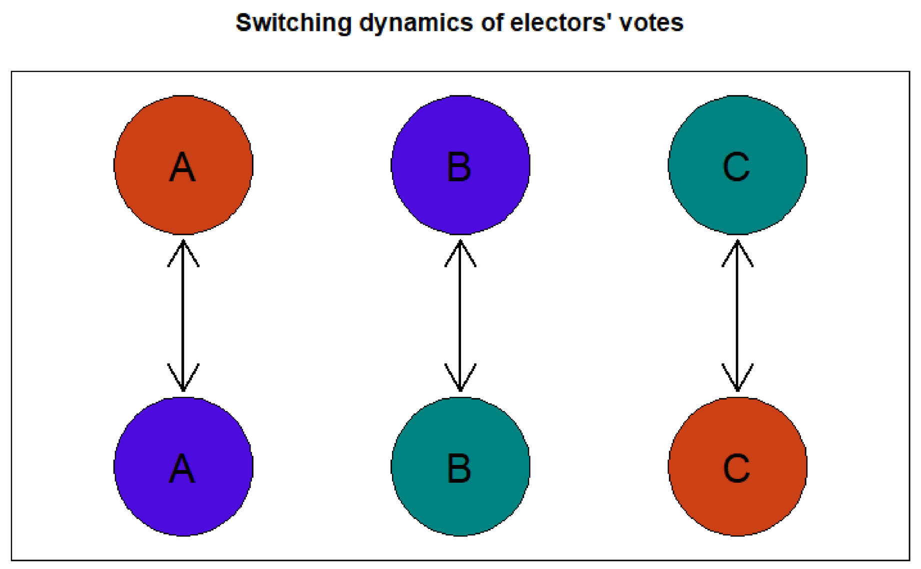
Iteration 1: Vote sincerely. $[v(\text{RED}), v(\text{BLUE}), v(\text{GREEN})] = [1, 1, 1]$



Iteration 2: Elector A can break a tie for BLUE. Elector B can break a tie for GREEN. And Elector C can break a tie for RED. So $[v(\text{RED}), v(\text{BLUE}), v(\text{GREEN})] = [1, 1, 1]$



Iteration 3: Elector A can break a tie for RED. Elector B can break a tie for BLUE. And Elector C can break a tie for GREEN. So $[v(\text{RED}), v(\text{BLUE}), v(\text{GREEN})] = [1, 1, 1]$. Repeat from Iteration 1

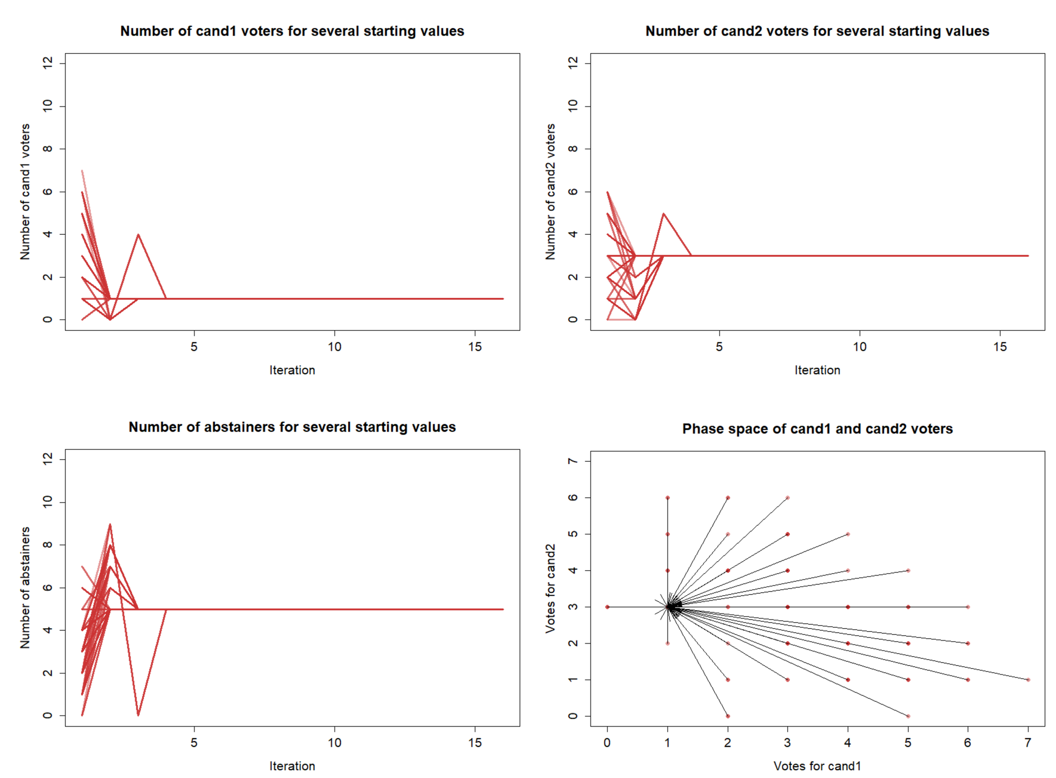


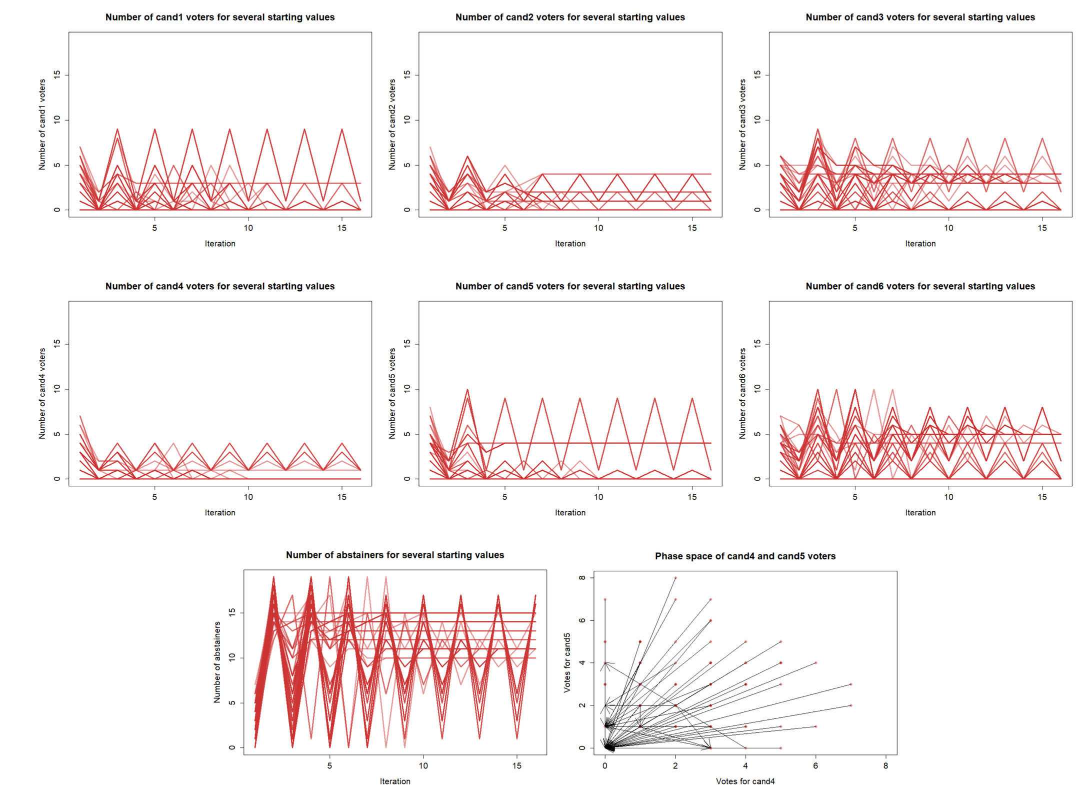
Solution concept: Equilibria and equilibrium cycles

Definition: A parameter, like a probability of voting for a given candidate, has reached an **equilibrium cycle** at iteration t if, $\forall t' > t$, it holds that $\forall t'' > t'$, $\exists k$ so $t'' - t' \equiv 0 \mod k \implies \rho_{t''} = \rho_{t'}$.

One static equilibrium concept that allows for sufficiently small alterations in numbers, if only because of the finite precision arithmetic in any computer model, is $\exists \epsilon$ sufficiently small so that $|\rho_{t'} - \rho_t| \leq \epsilon \ \forall t'$ with $t' > t$. With $\epsilon = 0$ we have a completely static equilibrium.

Examples of static equilibria compared to equilibrium cycles using a finite sample approximation of poisson voting games where electors begin by voting randomly:





Simple ruleset: Naïve Pivotality

A simple base case for electors (with sufficiently small positive costs) to estimate their pivotal probabilities:

↪ Electors signal their sincere voting intentions
↪ In the next iteration, electors check whether or not they would have been pivotal in the previous iteration
  – If an elector would have been pivotal in creating or breaking some tie, they assume that they will obtain their sincere probability for that vote choice with certainty
  – If an elector would not have been pivotal in creating or breaking some tie, they assume they will obtain no utility from that vote choice
↪ Electors signal their best response, which will be taken as the expected vote total in the next iteration

Properties of Naïve Pitovality:

Theorem: Any time we seed a system that does not produce a tie or a tie ± 1, we will expect a cycle.

Summary of proof: Any vector of aggregate vote totals which does not include a tie or a tie ± 1 will result in universal abstention in the next iteration.

Theorem: In a k-cyclic system using Naïve Pivotality, $k \in \mathbb{N}$, electors can only cycle through strategies with period 1 or k

Summary of proof: We can prove this using a contradiction, based on the observation that if a candidate has a nonzero vote total in iteration $t + 1$ they must have been in a first-place tie in iteration t. Non k-cyclers must violate payoff-maximizing behaviour.

Theorem: The unique stable state of the iterative system under Naïve Pivotality is the case in which every elector is voting in a tie or a tie ± 1. Substantively, this means that Cox's (1994) Duvergerian or non-Duvergerian results are the unique rest points of Naïve Pivotality electorates.

Summary of proof: We can prove this with the lemma that candidates' vote totals only increase if they were involved in a tie or a tie ± 1, and they cannot decrease in this case.

Threshold pivotality

In Naïve Pivotality, electors check whether or not they would have been pivotal in the previous iteration. If so, they vote. This is quite artificial.

However, all of my results are trivially extended to the situation where an elector votes iff they would have been within N votes of making or breaking a first-place tie, or within $N\%$ of the total votes. This is much more substantively satisfying: we can imagine someone voting if their party is within 5% of the closest opponent in the latest poll. Here, we have an equilibrium cycle whenever we seed a system that does not have a tie or a tie $\pm N$, and the Cox (1994) result remains the sole stable state, but with a more generous idea of a near-tie.

Deterministic models

Deterministic iterative voting game: Any situation in which each elector maps an aggregate vote count onto one unique strategy (a vote for a particular candidate, or abstention)

Properties of deterministic iterative voting games:

Theorem: The vote counts of a deterministic election game cannot wander forever without repetition. There must exist some k for which the vote counts of any deterministic election game are k-cyclic.

Summary of proof: To see this, enumerate the possible values that a given iterative election game could adopt, and note that each one must lead to another one with certainty. We can show that this provides a finite upper bound to the cyclicity of the system.

Theorem: If the aggregate vote count in iteration $t + 1$ equals the aggregate vote count in iteration t, then every elector has the same best response in $t + 1$ as they had in t.

Summary of proof: This is a direct result of the definition of a deterministic voting game.

Theorem: If the aggregate vote totals repeat with period k, then the electorate can only contain individuals whose cyclicity divide k

Summary of proof: We can show that the period of the system is the least common multiple of all the individual periods in the system.

Corollary: If the cyclicity of the electorate is prime, then every elector has exactly that cyclicity or 1.

Next steps

↪ Probabilistic iterative election models
↪ Empirics: checking for bouncing in longitudinal surveys
↪ Connections to sequential instead of simultaneous voting